Austrian-Polish Competition 1996

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–	Individual
1	Let $k \geq 1$ be a positive integer. Prove that there exist exactly 3^{k-1} natural numbers n with the following properties: (i) n has exactly k digits (in decimal representation), (ii) all the digits of n are odd, (iii) n is divisible by 5, (iv) the number $m = n/5$ has k odd digits
2	A convex hexagon $ABCDEF$ satisfies the following conditions: 1) $AB \parallel DE$, $BC \parallel EF$, and $CD \parallel FA$. 2) The distances between these pairs of parallel lines are the same. 3) $\angle FAB = \angle CDE = 90^\circ$ Prove that the diagonals BE and CF of the hexagon intersect with angle 45 degrees. • Thank you dear **Babis Stergiou** for your translation. :P
3	The polynomials $P_n(x)$ are defined by $P_0(x) = 0, P_1(x) = x$ and $$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \quad n \geq 2$$ For every natural number $n \geq 1$, find all real numbers x satisfying the equation $P_n(x) = 0$.
4	Real numbers x, y, z, t satisfy $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that $-1 \leq xy + yz + zt + tx \leq 0$.
5	A sphere S divides every edge of a convex polyhedron P into three equal parts. Show that there exists a sphere tangent to all the edges of P.
6	Given natural numbers $n > k > 1$, find all real solutions $x_1, ..., x_n$ of the system $$x_i^3(x_i^2 + x_{i+1}^2 + ... + x_{i+k-1}^2) = x_{i-1}^2$$ for 1 $\leq i \leq n$. Here $x_{n+i} = x_i$ for all$i$.
–	Team
7	Prove there are no such integers k, m which satisfy $k \geq 0, m \geq 0$ and $k! + 48 = 48(k+1)^m$.

8 Show that there is no polynomial $P(x)$ of degree 998 with real coefficients which satisfies $P(x^2+1) = P(x)^2 - 1$ for all x.

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9 For any triple (a, b, c) of positive integers, not all equal, We are given sufficiently many rectangular blocks of size $a \times b \times c$. We use these blocks to fill up a cubic box of edge 10.
(a) Assume we have used at least 100 blocks. Show that there are two blocks, one of which is a translate of the other.
(b) Find a number smaller than 100 (the smaller, the better) for which the above statement still holds.
