Austrian-Polish Competition 1997

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by parmenides51, ShahinBJK, EQSon, lehungvietbao

– Individual

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1 Let P be the intersection of lines l_1 and l_2. Let S_1 and S_2 be two circles externally tangent at P and both tangent to l_1, and let T_1 and T_2 be two circles externally tangent at P and both tangent to l_2.
Let A be the second intersection of S_1 and T_1, B that of S_1 and T_2, C that of S_2 and T_1, and D that of S_2 and T_2. Show that the points A, B, C, D are concyclic if and only if l_1 and l_2 are perpendicular.

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2 Each square of an $n \times m$ board is assigned a pair of coordinates (x, y) with $1 \leq x \leq m$ and $1 \leq y \leq n$. Let p and q be positive integers. A pawn can be moved from the square (x, y) to (x', y') if and only if $|x - x'| = p$ and $|y - y'| = q$. There is a pawn on each square. We want to move each pawn at the same time so that no two pawns are moved onto the same square. In how many ways can this be done?

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3 Numbers $\frac{49}{1}, \frac{49}{2}, ..., \frac{49}{97}$ are writen on a blackboard. Each time, we can replace two numbers (like a, b) with $2ab - a - b + 1$. After 96 times doing that prenominate action, one number will be left on the board. Find all the possible values fot that number.

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4 In a trapezoid $ABCD$ with $AB//CD$, the diagonals AC and BD intersect at point E. Let F and G be the orthocenters of the triangles EBC and EAD. Prove that the midpoint of GF lies on the perpendicular from E to AB.

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5 Let p_1, p_2, p_3, p_4 be four distinct primes. Prove that there is no polynomial $Q(x) = ax^3 + bx^2 + cx + d$ with integer coefficients such that $|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3$.

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6 Show that there is no integer-valued function on the integers such that $f(m+f(n)) = f(m)-n$ for all m, n.

– Team

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7 (a) Prove that $p^2 + q^2 + 1 > p(q + 1)$ for any real numbers p, q, .
(b) Determine the largest real constant b such that the inequality $p^2 + q^2 + 1 \geq bp(q + 1)$ holds for all real numbers p, q

(c) Determine the largest real constant c such that the inequality $p^2 + q^2 + 1 \geq cp(q + 1)$ holds for all integers p, q.

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8 Let X be a set with n elements. Find the largest number of subsets of X, each with 3 elements, so that no two of them are disjoint.

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9 Given a parallelepiped P, let V_P be its volume, S_P the area of its surface and L_P the sum of the lengths of its edges. For a real number $t \geq 0$, let P_t be the solid consisting of all points X whose distance from some point of P is at most t. Prove that the volume of the solid P_t is given by the formula $V(P_t) = V_P + S_P t + \frac{\pi}{4} L_P t^2 + \frac{4\pi}{3} t^3$.
