Austrian-Polish Competition 1999

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–	Individual
1	Find the number of 6-tuples $(A_1, A_2, ..., A_6)$ of subsets of $M = \{1, ..., n\}$ (not necessarily different) such that each element of M belongs to zero, three, or six of the subsets $A_1, ..., A_6$.
2	Find the best possible k, k' such that $$k < \frac{v}{v+w} + \frac{w}{w+x} + \frac{x}{x+y} + \frac{y}{y+z} + \frac{z}{z+v} < k'$$ for all positive reals v, w, x, y, z.
3	Given an integer $n \geq 2$, find all sustems of n functions $f_1, ..., f_n : R \to R$ such that for all $x, y \in R$ $f_1(x) - f_2(x)f_2(y) + f_1(y) = 0$ $f_2(x^2) - f_3(x)f_3(y) + f_2(y^2) = 0$ ... $f_n(x^n) - f_1(x)f_1(y) + f_n(y^n) = 0$
4	Three lines k, l, m are drawn through a point P inside a triangle ABC such that k meets AB at A_1 and AC at $A_2 \neq A_1$ and $PA_1 = PA_2$, l meets BC at B_1 and BA at $B_2 \neq B_1$ and $PB_1 = PB_2$, m meets CA at C_1 and CB at $C_2 \neq C_1$ and $PC_1 = PC_2$. Prove that the lines k, l, m are uniquely determined by these conditions. Find point P for which the triangles $AA_1A_2, BB_1B_2, CC_1C_2$ have the same area and show that this point is unique.
5	A sequence of integers (a_n) satisfies $a_{n+1} = a_n^3 + 1999$ for $n = 1, 2, ...$ Prove that there exists at most one n for which a_n is a perfect square.
6	Solve in the nonnegative real numbers the system of equations $x_n^2 + x_nx_{n-1} + x_{n-1}^4 = 1$ for $n = 1, 2, ..., 1999$, $x_0 = x_{1999}$.
–	Team
7	Find all pairs (x, y) of positive integers such that $x^{x+y} = y^{y-x}$.
8	Let P, Q, R be points on the same side of a line g in the plane. Let M and N be the feet of the perpendiculars from P and Q to g respectively. Point S lies between the lines PM and QN and satisfies and satisfies $PM = PS$ and $QN = QS$. The perpendicular bisectors of SM and SN meet in a point R. If the line RS intersects the circumcircle of triangle PQR again at T, prove that S is the midpoint of RT. Austrian-Polish 1999

9 A point in the cartesian plane with integer coordinates is called a lattice point. Consider the following one player game. A finite set of selected lattice points and finite set of selected segments is called a position in this game if the following hold:
(i) The endpoints of each selected segment are lattice points;
(ii) Each selected segment is parallel to a coordinate axis or to one of the lines $y = \pm x$,
(iii) Each selected segment contains exactly five lattice points, all of which are selected,
(iv) Every two selected segments have at most one common point.
A move in this game consists of selecting a lattice point and a segment such that the new set of selected lattice points and segments is a position. Prove or disprove that there exists an initial position such that the game can have infinitely many moves.