Austrian-Polish Competition 2000

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– Individual

1 Find all polynomials $P(x)$ with real coefficients having the following property: There exists a positive integer n such that the equality

$$\sum_{k=1}^{2n+1}(-1)^k\left[\frac{k}{2}\right]P(x+k)=0$$

holds for infinitely many real numbers x.

2 In a unit cube, CG is the edge perpendicular to the face $ABCD$. Let O_1 be the incircle of square $ABCD$ and O_2 be the circumcircle of triangle BDG. Determine min$\{XY|X\in O_1, Y\in O_2\}$.

3 For each integer $n\geq 3$ solve in real numbers the system of equations:

$$\begin{cases} x_1^3=x_2+x_3+1 \\ ... \\ x_{n-1}^3=x_n+x_1+1 \\ x_n^3=x_1+x_2+1 \end{cases}$$

4 Find all positive integers N having only prime divisors $2,5$ such that $N+25$ is a perfect square.

5 For which integers $n\geq 5$ is it possible to color the vertices of a regular$n$-gon using at most 6 colors in such a way that any 5 consecutive vertices have different colors?

6 Consider the solid Q obtained by attaching unit cubes $Q_1 ... Q_6$ at the six faces of a unit cube Q. Prove or disprove that the space can be filled up with such solids so that no two of them have a common interior point.

– Team

7 Triangle $A_0B_0C_0$ is given in the plane. Consider all triangles ABC such that:
(i) The lines AB,BC,CA pass through C_0,A_0,B_0, respectvely,
(ii) The triangles ABC and $A_0B_0C_0$ are similar.
Find the possible positions of the circumcenter of triangle ABC.

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8 In the plane are given 27 points, no three of which are collinear. Four of this points are vertices of a unit square, while the others lie inside the square. Prove that there are three points in this set forming a triangle with area not exceeding $1/48$.

9 If three nonnegative reals a, b, c satisfy $a+b+c=1$, prove that
$2 \le \left(1-a^2\right)^2+\left(1-b^2\right)^2+\left(1-c^2\right)^2 \le (1+a)(1+b)(1+c)$.

10 The plan of the castle in Baranow Sandomierski can be presented as the graph with 16 vertices on the picture.
A night guard plans a closed round along the edges of this graph.
(a) How many rounds passing through each vertex exactly once are there? The directions are irrelevant.
(b) How many non-selfintersecting rounds (taking directions into account) containing each edge of the graph exactly once are there?
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