Problems from the 2017-2018 SDPC. Middle School division does 1,2,3,4,5, High School division does 3,4,5,6,7.

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by mira74

1 Lucky starts doodling on a 5×5 Bingo board. He puts his pencil at the center of the upper-left square (marked by) and draws a continuous doodle ending on the Free Space, never going off the board or through a corner of a square. (See Figure 1.)

(a) Is it possible for Luckys doodle to visit all squares exactly once? (The starting and ending squares are considered visited.)
(b) Is it possible for Luckys doodle to visit all squares exactly twice?

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2 Call a quadratic *invasive* if it has 2 distinct real roots. Let P be a quadratic polynomial with real coefficients. Prove that $P(x)$ is invasive **if and only if** there exists a real number $c \neq 0$ such that $P(x) + P(x - c)$ is invasive.

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3 Let $n > 2$ be a fixed positive integer. For a set S of n points in the plane, let $P(S)$ be the set of perpendicular bisectors of pairs of distinct points in S. Call set S *complete* if no two (distinct) pairs of points share the same perpendicular bisector, and every pair of lines in $P(S)$ intersects. Let $f(S)$ be the number of distinct intersection points of pairs of lines in $P(S)$.

(a) Find all complete sets S such that $f(S) = 1$.
(b) Let S be a complete set with n points. Show that if $f(S) > 1$, then $f(S) \geq n$.

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4 Call a positive rational number in simplest terms *coddly* if its numerator and denominator are both odd. Consider the equation

$$2017 = x_1 \,\square\, x_2 \,\square\, x_3 \;\ldots\; \square\, x_{2016} \,\square\, x_{2017},$$

where there are 2016 boxes. We fill up the boxes randomly with the operations $+$, $-$, and $\times$. Compute the probability that there exists a solution in **distinct** coddly numbers $(x_1, x_2, \ldots x_{2017})$ to the resulting equation.

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5 Given positive real numbers a, b, c such that $abc = 1$, find the maximum possible value of

$$\frac{1}{(4a+4b+c)^3} + \frac{1}{(4b+4c+a)^3} + \frac{1}{(4c+4a+b)^3}.$$

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6 Let ABC be an acute triangle with circumcenter O. Let the parallel to BC through A intersect line BO at B_A and CO at C_A. Lines B_AC and BC_A intersect at A'. Define B' and C' similarly.

(a) Prove that the the perpendicular from A' to BC, the perpendicular from B' to AC, and C' to AB are concurrent.
(b) Prove that likes AA', BB', and CC' are concurrent.

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7 Let $n > 1$ be a fixed integer. On an infinite row of squares, there are n stones on square 1 and no stones on squares $2, 3, 4, \ldots$. Curious George plays a game in which a *move* consists of taking two adjacent piles of sizes a and b, where $a - b$ is a nonzero even integer, and transferring stones to equalize the piles (so that both piles have $\frac{a+b}{2}$ stones). The game ends when no more moves can be made. George wants to analyze the number of moves it takes to end the game.

(a) Suppose George wants to end the game as quickly as possible. How many moves will it take him?
(b) Suppose George wants to end the game as slowly as possible. Show that for all $n > 2$, the game will end after at most $\frac{3}{16}n^2$ moves.

Scoring note: For part (b), partial credit will be awarded for correct proofs of weaker bounds, eg. $\frac{1}{4}n^2$, n^k, or k^n (for some $k \geq 2$).
