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W. 1 The Pell numbers P_n satisfy $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Find

$$\sum_{n=1}^{\infty} \left(\tan^{-1}\frac{1}{P_{2n}} + \tan^{-1}\frac{1}{P_{2n+2}}\right)\tan^{-1}\frac{2}{P_{2n+1}}$$

W. 2 If $0 < a \leq c \leq b$ then

$$\frac{(b^{30}-a^{30})(b^{30}-c^{30})}{36b^{10}} \leq \frac{(b^{25}-a^{25})(b^{25}-c^{25})}{25} \leq \frac{(b^{30}-a^{30})(b^{30}-c^{30})}{36(ac)^{10}}$$

W. 3 Compute

$$\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x + 1 - x^2}{(1 + x\sin x)\sqrt{1-x^2}}dx$$

W. 4 If $x, y, z, t > 1$ then:

$$\left(\log_{zxt} x\right)^2 + \left(\log_{xyt} y\right)^2 + \left(\log_{xyz} z\right)^2 + \left(\log_{yzt} t\right)^2 > \frac{1}{4}$$

W. 5 Let $n \geq 1$. Find a set of distincts real numbers $(x_j)_{1\leq j\leq n}$ such that for any bijections $f : \{1, 2, \cdots, n\}^2 \to \{1, 2, \cdots, n\}^2$ the matrix $\left(x_{f(i,j)}\right)_{1\leq i,j\leq n}$ is invertible.

W. 6 Compute

$$\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(1+\ln x)\cos x + x\sin x\ln x}{\cos^2 x + x^2\ln^2 x}dx$$

W. 7 If

$$\Omega_n = \sum_{k=1}^{n}\left(\int\limits_{-\frac{1}{k}}^{\frac{1}{k}} (2x^{10} + 3x^8 + 1)\cos^{-1}(kx)dx\right)$$

Then find

$$\Omega = \lim_{n\to\infty} \left(\Omega_n - \pi H_n\right)$$

W. 8 Let $(a_n)_{n\geq 1}$ be a positive real sequence given by $a_n = \sum\limits_{k=1}^{n} \frac{1}{k}$. Compute

$$\lim_{n\to\infty} e^{-2a_n} \sum_{k=1}^{n} \left\lfloor \left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!} \right)^2 \right\rfloor$$

where we denote by $\lfloor x \rfloor$ the integer part of x.

W. 9 Let $\alpha > 0$ be a real number. Compute the limit of the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_n = \begin{cases} \sum\limits_{k=1}^{n} \sinh\left(\frac{k}{n^2}\right), & \text{when } n > \frac{1}{\alpha} \\ 0, & \text{when } n \leq \frac{1}{\alpha} \end{cases}$$

W. 10 If $si(x) = -\int\limits_{x}^{\infty} \left(\frac{\sin t}{t}\right) dt; x > 0$ then

$$\int\limits_{e}^{e^2} \left(\frac{1}{x} \left(si\left(e^4 x\right) - si\left(e^3 x\right) \right) \right) dx = \int\limits_{3}^{e^4} \left(\frac{1}{x} \left(\text{si}\left(e^2 x\right) - si\left(ex\right) \right) \right) dx$$

W. 11 Let $(s_n)_{n\geq 1}$ be a sequence given by $s_n = -2\sqrt{n} + \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}}$ with $\lim\limits_{n\to\infty} s_n = s =$Ioachimescu constant and $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ be a positive real sequences such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{n a_n} = a \in \mathbb{R}_+^*, \lim_{n\to\infty} \frac{b_{n+1}}{b_n \sqrt{n}} = b \in \mathbb{R}_+^*$$

Compute

$$\lim_{n\to\infty} \left(1 + e^{s_n} - e^{s_{n+1}}\right)^{\sqrt[n]{a_n b_n}}$$

W. 12 If $0 < a < b$ then:

$$\frac{\int\limits_{a}^{\frac{a+b}{2}} \left(\tan^{-1} t\right) dt}{\int\limits_{a}^{b} \left(\tan^{-1} t\right) dt} < \frac{1}{2}$$

W. 13 Let a, b and c be complex numbers such that $abc = 1$. Find the value of the cubic root of

$$\begin{vmatrix} b+n^3c & n(c-b) & n^2(b-c) \\ n^2(c-a) & c+n^3a & n(a-c) \\ n(b-a) & n^2(a-b) & a+n^3b \end{vmatrix}$$

W. 14 If $a, b, c > 0$; $ab + bc + ca = 3$ then:

$$4\left(\tan^{-1}2\right)\left(\tan^{-1}\left(\sqrt[3]{abc}\right)\right) \leq \pi \tan^{-1}\left(1+\sqrt[3]{abc}\right)$$

W. 15 It is possible to partition the set $\{100, 101, \cdots, 1000\}$ into two subsets so that for any two distinct elements x and y belonging to the same subset $\sqrt[3]{x+y}$ is irrational?

W. 16 If $f : [a, b] \to (0, \infty)$; $0 < a \leq b$; f derivable; f' continuous then:

$$\int_a^b \frac{f'(x)\sqrt{f(x)}}{f^3(x)+1} \leq \tan^{-1}\left(\frac{f(b)-f(a)}{1+f(a)f(b)}\right)$$

W. 17 Let $f_n = \left(1+\frac{1}{n}\right)^n ((2n-1)!F_n)^{\frac{1}{n}}$. Find $\lim\limits_{n\to\infty}(f_{n+1}-f_n)$ where F_n denotes the nth Fibonacci number (given by $F_0 = 0$, $F_1 = 1$, and by $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$

W. 18 Let $\{c_k\}_{k\geq 1}$ be a sequence with $0 \leq c_k \leq 1$, $c_1 \neq 0$, $\alpha > 1$. Let $C_n = c_1 + \cdots + c_n$. Prove

$$\lim_{n\to\infty}\frac{C_1^\alpha + \cdots + C_n^\alpha}{(C_1 + \cdots + C_n)^\alpha} = 0$$

W. 19 Let $\{F_n\}_{n\in\mathbb{Z}}$ and $\{L_n\}_{n\in\mathbb{Z}}$ denote the Fibonacci and Lucas numbers, respectively, given by

$$F_{n+1} = F_n + F_{n-1} \text{ and } L_{n+1} = L_n + L_{n-1} \text{ for all } n \geq 1$$

with $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$. Prove that for integers $n \geq 1$ and $j \geq 0$

- $\sum\limits_{k=1}^n F_{k\pm j}L_{k\mp j} = F_{2n+1} - 1 + \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{\pm j}\,F_{\pm 2j}, & \text{if } n \text{ is odd} \end{cases}$
- $\sum\limits_{k=1}^n F_{k+j}F_{k-j}L_{k+j}L_{k-j} = \frac{F_{4n+2}-1-nL_{4j}}{5}$

W. 20

- Let G be a $(4,4)$ unoriented graph, 2-regulate, containing a cycle with the length 3. Find the characteristic polynomial $P_G(\lambda)$, its spectrum $Spec(G)$ and draw the graph G.
- Let G' be another 2-regulate graph, having its characteristic polynomial $P_{G'}(\lambda)=\lambda^4-4\lambda^2+\alpha, \alpha\in\mathbb{R}$. Find the spectrum $Spec(G')$ and draw the graph G'.
- Are the graphs G and G' cospectral or isomorphic?

W. 21 Let f be a continuously differentiable function on $[0,1]$ and $m\in\mathbb{N}$. Let $A=f(1)$ and let $B=\int\limits_0^1 x^{-\frac{1}{m}}f(x)dx$. Calculate

$$\lim_{n\to\infty} n\left(\int\limits_0^1 f(x)dx-\sum_{k=1}^{n}\left(\frac{k^m}{n^m}-\frac{(k-1)^m}{n^m}\right)f\left(\frac{(k-1)^m}{n^m}\right)\right)$$

in terms of A and B.

W. 22 Let A and B the series:

$$A=\sum_{n=1}^{\infty}\frac{C_{2n}^1}{C_{2n}^0+C_{2n}^1+\cdots+C_{2n}^{2n}},\quad B=\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{5}{2}\right)}$$

Study if $\frac{A}{B}$ is irrational number.

W. 23 If b, c are the legs, and a is the hypotenuse of a right triangle, prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 5+3\sqrt{2}$$

W. 24 If a, b, $c>0$, prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{a+b}{a+b+2c}+\frac{b+c}{2a+b+c}+\frac{c+a}{a+2b+c}$$

W. 25 Let x_i, y_i, z_i, $w_i\in\mathbb{R}^+, i=1,2,\cdots n$, such that

$$\sum_{i=1}^{n}x_i=nx,\ \sum_{i=1}^{n}y_i=ny,\ \sum_{i=1}^{n}w_i=nw$$

$$\Gamma\left(z_i\right)\geq\Gamma\left(w_i\right),\ \sum_{i=1}^{n}\Gamma\left(z_i\right)=n\Gamma^*(z)$$

Then

$$\sum_{i=1}^{n} \frac{\left(\Gamma\left(x_i\right)+\Gamma\left(y_i\right)\right)^2}{\Gamma\left(z_i\right)-\Gamma\left(w_i\right)} \geq n \frac{\left(\Gamma\left(x\right)+\Gamma\left(y\right)\right)^2}{\Gamma^*\left(z\right)-\Gamma\left(w\right)}$$

W. 26 Let $n \in \mathbb{N}$, $n \geq 2$, $a_1, a_2, \cdots, a_n \in \mathbb{R}$ and $a_n = max\{a_1, a_2, \cdots, a_n\}$

-If t_k, $t'_k \in \mathbb{R}$, $k \in \{1, 2, \cdots, n\}$, $t_k \leq t'_k$, for any $k \in \{1, 2, \cdots, n-1\}$ and

$$\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} t'_k$$

Prove that

$$\sum_{k=1}^{n} t_k a_k \geq \sum_{k=1}^{n} t'_k a_k$$

- If b_k, $c_k \in \mathbb{R}_+^*$, $k \in \{1, 2, \cdots, n\}$, $b_k \leq c_k$ for any $k \in \{1, 2, \cdots, k-1\}$ and

$$b_1 b_2 \cdots b_n = c_1 c_2 \cdots c_n$$

Prove that

$$\prod_{k=1}^{n} b_k^{a_k} \geq \prod_{k=1}^{n} c_k^{a_k}$$

W. 27 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(-x) + \int\limits_0^x t f(x-t) dt = x,\ \forall\, x \in \mathbb{R}$$

W. 28 In a room, we have 2019 aligned switches, connected to 2019 light bulbs, all initially switched on. Then, 2019 people enter the room one by one, performing the operation: The first, uses all the switches; the second, every second switch; the third, every third switch, and so on. How many lightbulbs remain switched on, after all the people entered ?

W. 29 Prove that

$$\int\limits_0^{\infty} e^{3t} \frac{4e^{4t}(3t-1) + 2e^{2t}(15t-17) + 18(1-t)}{(1+e^{4t}-e^{2t})^2} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} - 10$$

W. 30

- Prove that

$$\lim_{n\to\infty}\left(n+\frac{1}{4}-\zeta(3)-\zeta(5)-\cdots-\zeta(2n+1)\right)=0$$

- Calculate

$$\sum_{n=1}^{\infty}\left(n+\frac{1}{4}-\zeta(3)-\zeta(5)-\cdots-\zeta(2n+1)\right)$$

W. 31 Let $a,b\in\Gamma$, $a<b$ and the differentiable function $f:[a,b]\to\Gamma$, such that $f(a)=a$ and $f(b)=b$. Prove that

$$\int\limits_a^b \left(f'(x)\right)^2 dx\geq b-a$$

W. 32 Let u_k, v_k, a_k and b_k be non-negative real sequences such as $u_k>a_k$ and $v_k>b_k$, where $k=1,2,\cdots,n$. If $0<m_1\leq u_k\leq M_1$ and $0<m_2\leq v_k\leq M_2$, then

$$\sum_{k=1}^{n}(lu_kv_k-a_kb_k)\geq\left(\sum_{k=1}^{n}\left(u_k^2-a_k^2\right)\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}\left(v_k^2-b_k^2\right)\right)^{\frac{1}{2}}$$

where

$$l=\frac{M_1M_2+m_1m_2}{2\sqrt{m_1M_1m_2M_2}}$$

W. 33 Let $0<\frac{1}{q}\leq\frac{1}{p}<1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let u_k, v_k, a_k and b_k be non-negative real sequences such as $u_k^2>a_k^p$ and $v_k>b_k^q$, where $k=1,2,\cdots,n$. If $0<m_1\leq u_k\leq M_1$ and $0<m_2\leq v_k\leq M_2$, then

$$\left(\sum_{k=1}^{n}\left(l^p\left(u_k+v_k\right)^2-\left(a_k+b_k\right)^p\right)\right)^{\frac{1}{p}}\geq\left(\sum_{k=1}^{n}\left(u_k^2-a_k^p\right)\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}\left(v_k^2-b_k^p\right)\right)^{\frac{1}{p}}$$

where

$$l=\frac{M_1M_2+m_1m_2}{2\sqrt{m_1M_1m_2M_2}}$$

W. 34 Let a, b, c be positive real numbers and let m, n $(m\geq n)$ be positive integers. Prove that

$$\frac{a^{n-1}b^{n-1}c^{m-n-1}}{a^{m+n}+b^{m+n}+a^nb^nc^{m-n}}+\frac{b^{n-1}bc^{n-1}a^{m-n-1}}{b^{m+n}+c^{m+n}+b^nc^na^{m-n}}+\frac{c^{n-1}a^{n-1}b^{m-n-1}}{c^{m+n}+a^{m+n}+c^na^nb^{m-n}}\leq\frac{1}{abc}$$

W. 35 Calculate

$$\lim_{n\to\infty} \frac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+\frac{1}{2}}}$$

W. 36 For any $a, b, c > 0$ and for any $n \in \mathbb{N}^*$, prove the inequality

$$(a-b)\left(\frac{a}{b}\right)^n + (b-c)\left(\frac{b}{c}\right)^n + (c-a)\left(\frac{c}{a}\right)^n \geq (a-b)\frac{a}{b} + (b-c)\frac{b}{c} + (c-a)\frac{c}{a}$$

W. 37 For real $a > 1$ find

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=2}^{n}\left(a - a^{\frac{1}{k}}\right)}$$

W. 38 Let a, b, c be the sides of an acute triangle $\triangle ABC$, then for any $x, y, z \geq 0$, such that $xy + yz + zx = 1$ holds inequality:

$$a^2x + b^2y + c^2z \geq 4F$$

where F is the area of the triangle $\triangle ABC$

W. 39 Let u, v, w complex numbers such that: $u + v + w = 1$, $u^2 + v^2 + w^2 = 3$, $uvw = 1$. Prove that

- u, v, w are distinct numbers two by two
- If $S(k) = u^k + v^k + w^k$, then $S(k)$ is an odd natural number
- The expression

$$\frac{u^{2n+1} - v^{2n+1}}{u - v} + \frac{v^{2n+1} - w^{2n+1}}{v - w} + \frac{w^{2n+1} - u^{2n+1}}{w - u}$$

is an integer number.

W. 40 Let f_n be nth Fibonacci number defined by recurrence $f_{n+1} - f_n - f_{n-1} = 0$, $n \in \mathbb{N}$ and initial conditions $f_0 = 0$, $f_1 = 1$. Prove that for any $n \in \mathbb{N}$

$$(n-1)(n+1)(2nf_{n+1} - (n+6)f_n)$$

is divisible by 150 for any $n \in \mathbb{N}$.

W. 41 For $n \in \mathbb{N}$, consider in $\mathbb{R}^3$ the regular tetrahedron with vertices $O(0,0,0)$, $A(n, 9n, 4n)$, $B(9n, 4n, n)$ and $C(4n, n, 9n)$. Show that the number N of points (x, y, z), $[x, y, z \in \mathbb{Z}]$ inside or on the boundary of the tetrahedron $OABC$ is given by

$$N = \frac{343n^3}{3} + \frac{35n^2}{2} + \frac{7n}{6} + 1$$

W. 42 For p, q, l strictly positive real numbers, consider the following problem: for $y \geq 0$ fixed, determine the values $x \geq 0$ such that $x^p - lx^q \leq y$. Denote by $S(y)$ the set of solutions of this problem. Prove that if one has $p < q$, $\epsilon \in (0, l^{\frac{1}{p-q}})$, $0 \leq x \leq \epsilon$ and $x \in S(y)$, then

$$x \leq ky^{\delta}, \text{ where } k = \epsilon\left(\epsilon^p - l\epsilon^q\right)^{-\frac{1}{p}} \text{ and } \delta = \frac{1}{p}$$

W. 43 Consider the sequence of polynomials $P_0(x) = 2$, $P_1(x) = x$ and $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$ for $n \geq 2$. Let x_n be the greatest zero of P_n in the the interval $|x| \leq 2$. Show that

$$\lim_{n\to\infty} n^2\left(4 - 2\pi + n^2\int\limits_{x_n}^{2} P_n(x)dx\right) = 2\pi - 4 - \frac{\pi^3}{12}$$

W. 44 We consider a natural number n, $n \geq 2$ and the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & 1 & 2 & \cdots & n-1 \\ n-1 & n & 1 & \cdots & n-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & 3 & 4 & \cdots & 1 \end{pmatrix}$$

Show that

$$\epsilon^n det\left(I_n - A^{2n}\right) + \epsilon^{n-1} det\left(\epsilon I_n - A^{2n}\right) + \epsilon^{n-2} det\left(\epsilon^2 I_n - A^{2n}\right) + \cdots + det\left(\epsilon^n I_n - A^{2n}\right)$$

$$= n(-1)^{n-1}\left[\frac{n^n(n+1)}{2}\right]^{2n^2-4n}\left(1 + (n+1)^{2n}\left(2n + (-1)^n\binom{2n}{n}\right)\right)$$

where $\epsilon \in \mathbb{C}\backslash\mathbb{R}$, $\epsilon^{n+1} = 1$

W. 45 Consider the complex numbers $a_1, a_2, \cdots, a_n$, $n \geq 2$. Which have the following properties:

- $|a_i| = 1 \; \forall \; i = 1, 2, \cdots, n$
- $\sum\limits_{k=1}^{n} arg(a_k) \leq \pi$

Show that the inequality

$$\left(n^2 \cot\left(\frac{\pi}{2n}\right)\right)^{-1}\left|\sum_{k=0}^{n}(-1)^k\left[3n^2 - (8k+5)n + 4k(k+1)\sigma_k\right]\right| \geq \sqrt{\left(1 + \frac{1}{n}\right)^2 \cot^2\left(\frac{\pi}{2n}\right)} + 16\left|\sum_{k=0}^{n}(-1)^k\sigma_k\right|$$

where $\sigma_0 = 1$, $\sigma_k = \sum\limits_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} a_{i_1}a_{i_2}\cdots a_{i_k}$, $\forall \; k = 1, 2, \cdots, n$

W. 46 Let $x, y, z > 0$ such that $x^2 + y^2 + z^2 = 3$. Then

$$x^3 \tan^{-1}\frac{1}{x} + y^3 \tan^{-1}\frac{1}{y} + z^3 \tan^{-1}\frac{1}{z} < \frac{\pi\sqrt{3}}{2}$$

W. 47

- If $a, b, c, d > 0$, show inequality:

$$\left(\tan^{-1}\left(\frac{ad-bc}{ac+bd}\right)\right)^2 \geq 2\left(1 - \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}\right)$$

- Calculate

$$\lim_{n\to\infty} n^{\alpha}\left(n - \sum_{k=1}^{n} \frac{n^{+}k^2-k}{\sqrt{(n^2+k^2)(n^2+(k-1)^2)}}\right)$$

where $\alpha \in \mathbb{R}$

W. 48 Let $f : (0, +\infty) \to \mathbb{R}$ a convex function and $\alpha, \beta, \gamma > 0$. Then

$$\frac{1}{6\alpha}\int_0^{6\alpha} f(x)dx + \frac{1}{6\beta}\int_0^{6\beta} f(x)dx + \frac{1}{6\gamma}\int_0^{6\gamma} f(x)dx$$

$$\geq \frac{1}{3\alpha+2\beta+\gamma}\int_0^{3\alpha+2\beta+\gamma} f(x)dx + \frac{1}{\alpha+3\beta+2\gamma}\int_0^{\alpha+3\beta+2\gamma} f(x)dx$$

$$+ \frac{1}{2\alpha+\beta+3\gamma}\int_0^{2\alpha+\beta+3\gamma} f(x)dx$$

W. 49 Let $a, b, c \in (0, +\infty)$. Then the following inequality is true:

$$\sqrt{(a+b)(b+c)}+\sqrt{(b+c)(c+a)}+\sqrt{(c+a)(a+b)}+a+b+c \leq (ab+bc+ca)\left(\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\right)$$

W. 50 Let $x, y, z > 0$, $\lambda \in (-\infty, 0) \cup (1, +\infty)$ such that $x + y + z = 1$. Then

$$\sum_{cyc} x^{\lambda}y^{\lambda} \sum_{cyc} \frac{1}{(x+y)^{2\lambda}} \geq 9\left(\frac{1}{4} - \frac{1}{9}\sum_{cyc}\frac{1}{(x+1)^2}\right)^{\lambda}$$

W. 51 Let a, b, c, d, e be real strictly positive real numbers such that $abcde = 1$. Then is true the following inequality:

$$\frac{de}{a(b+1)} + \frac{ea}{b(c+1)} + \frac{ab}{c(d+1)} + \frac{bc}{d(e+1)} + \frac{cd}{e(a+1)} \geq \frac{5}{2}$$

W. 52 Let $f : \mathbb{R} \to \mathbb{R}$ a periodic and continue function with period T and $F : \mathbb{R} \to \mathbb{R}$ antiderivative of f. Then

$$\int\limits_0^T \left[F(nx) - F(x) - f(x)\frac{(n-1)T}{2}\right] dx = 0$$

W. 53 Compute

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\sqrt[n+k+1]{n+1} - \sqrt[n+k]{n}}{\sqrt[n+k]{n+1} - \sqrt[n+k]{n}}$$

W. 54 Let $x_1, x_2, \geq, x_n$ be a positive numbers, $k \geq 1$. Then the following inequality is true:

$$\left(x_1^k + x_2^k + \cdots + x_n^k\right)^{k+1} \geq \left(x_1^{k+1} + x_2^{k+1} \cdots + x_n^{k+1}\right)^k + 2\left(\sum_{1\leq i<j\leq n} x_i^k x_j\right)^k$$

W. 55 Let $a_1, a_2, \cdots, a_n$ be n positive numbers such that $\sum\limits_{i=1}^{n} \sqrt{a_i} = \sqrt{n}$. Then

$$\prod_{i=1}^{n-1} \left(1 + \frac{1}{a_i}\right)^{a_{i+1}} \left(1 + \frac{1}{a_n}\right)^{a_1} \geq 1 + \frac{n}{\sum\limits_{i=1}^{n} a_i}$$

W. 56 Let $f, g, h : [a, b] \to \mathbb{R}$, three integrable functions such that:

$$\int\limits_a^b fg dx = \int\limits_a^b gh dx = \int\limits_a^b hf dx = \int\limits_a^b g^2 dx \int\limits_a^b h^2 dx = 1$$

Then

$$\int\limits_a^b g^2 dx = \int\limits_a^b h^2 dx = 1$$

W. 57 Let be $x_1 = \frac{1}{\sqrt[n+1]{n!}}$ and $x_2 = \frac{1}{\sqrt[n+1]{(n-1)!}}$ for all $n \in \mathbb{N}^*$ and $f : \left(\frac{1}{\sqrt[n+1]{(n+1)!}}, 1\right] \to \mathbb{R}$ where

$$f(x) = \frac{n+1}{x\ln(n+1)! + (n+1)\ln(x^x)}$$

Prove that the sequence $(a_n)_{n\geq 1}$ when $a_n = \int\limits_{x_1}^{x_2} f(x)dx$ is convergent and compute

$$\lim_{n\to\infty} a_n$$

W. 58 In the $[ABCD]$ tetrahedron having all the faces acute angled triangles, is denoted by r_X, R_X the radius lengths of the circle inscribed and circumscribed respectively on the face opposite to the $X \in \{A, B, C, D\}$ peak, and with R the length of the radius of the sphere circumscribed to the tetrahedron. Show that inequality occurs

$$8R^2 \geq (r_A + R_A)^2 + (r_B + R_B)^2 + (r_C + R_C)^2 + (r_D + R_D)^2$$

W. 59 In the any $[ABCD]$ tetrahedron we denote with α, β, γ the measures, in radians, of the angles of the three pairs of opposite edges and with r, R the lengths of the rays of the sphere inscribed and respectively circumscribed the tetrahedron. Demonstrate inequality

$$\left(\frac{3r}{R}\right)^3 \leq \sin\frac{\alpha+\beta+\gamma}{3}$$

(A refinement of inequality $R \geq 3r$).

W. 60 In all tetrahedron $ABCD$ holds

- $(n(n+2))^{\frac{1}{n}} \sum\limits_{cyc} \left(\frac{(h_a-r)^2}{(h_a^n-r^n)(h_a^{n+2}-r^{n+2})}\right)^{\frac{1}{n}} \leq \frac{1}{r^2}$
- $(n(n+2))^{\frac{1}{n}} \sum\limits_{cyc} \left(\frac{(r_a-r)^2}{(r_a^n-r^n)(r_a^{n+2}-r^{n+2})}\right)^{\frac{1}{n}} \leq \frac{1}{r^2}$

for all $n \in \mathbb{N}^*$

W. 61 If a, b, $c \in \mathbb{R}$ then

$$\sum_{cyc} \sqrt{(c+a)^2b^2 + c^2a^2} + \sqrt{5}\left|\sum_{cyc}\sqrt{ab}\right| \geq \sum_{cyc}\sqrt{(ab+2bc+ca)^2 + (b+c)^2a^2}$$

W. 62 Prove that

$$\int_0^{\frac{\pi}{2}} (\cos x)^{1+\sqrt{2n+1}} dx \le \frac{2^{n-1} n! \sqrt{\pi}}{\sqrt{2(2n+1)!}}$$

for all $n \in \mathbb{N}^*$

W. 63 If $b_k \ge a_k \ge 0$ $(k = 1, 2, 3)$ and $\alpha \ge 1$ then

$$(\alpha+3)\sum_{cyc}(b_1 - a_1)\left((b_2+b_3)^{\alpha+2} + (a_2+a_3)^{\alpha+2} - (a_2+b_3)^{\alpha+1} - (b_2+a_3)^{\alpha+1}\right)$$

$$\le (\alpha+2)(\alpha+3)\sum_{cyc}(b_1-a_1)(b_2-a_2)(b_3^{\alpha+1} - a_3^{\alpha+1})$$

$$+(b_3+b_2+a_1)^{\alpha+3} + (b_3+a_2+a_1)^{\alpha+3} + (a_3+b_2+a_1)^{\alpha+3} + (a_3+a_2+b_1)^{\alpha+3}$$

$$-(b_3+b_2+b_1)^{\alpha+3} - (b_3+a_2+a_1)^{\alpha+3} - (a_3+b_2+b_1)^{\alpha+3} - (a_3+a_2+a_1)^{\alpha+3}$$

W. 64 Prove that exist different natural numbers x, y, z, t for which

$$256 \times 2019^{180n+1} = 2x^9 - 2y^6 + z^5 - t^4$$

for all $n \in \mathbb{N}^*$

W. 65 If $a, b, c \ge 1$; $y \ge x \ge 1$; $p, q, r > 0$ then

$$\left(\frac{1+y\left(a^p b^q c^r\right)^{\frac{1}{p+q+r}}}{1+x\left(a^p b^q c^r\right)^{\frac{1}{p+q+r}}}\right)^{\frac{p+q+r}{\left(a^p b^q c^r\right)^{\frac{1}{p+q+r}}}} \left(\frac{1+xa}{1+ya}\right)^{\frac{p}{a}} \left(\frac{1+xb}{1+yb}\right)^{\frac{q}{b}} \left(\frac{1+xc}{1+yc}\right)^{\frac{r}{c}}$$

$$\ge \prod_{cyc}\left(\frac{1+y\left(a^p b^q\right)^{\frac{1}{p+q}}}{1+x\left(a^p b^q\right)^{\frac{1}{p+q}}}\right)^{\frac{p+q}{\left(a^p b^q\right)^{\frac{1}{p+q}}}}$$

W. 66 If $0 < a \le b$ then

$$\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2(b^2-a^2)}{(a^2+2)(b^2+2)}\right) \le \int_a^b \frac{(x^2+1)(x^2+x+1)}{(x^3+x^2+1)(x^3+x+1)}dx \le \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{(b-a)\sqrt{3}}{a+b+2(1+ab)}\right)$$

W. 67 Denote T the Toricelli point of the triangle ABC. Prove that

$$AB^2 \times BC^2 \times CA^2 \ge 3(TA^2 \times TB + TB^2 \times TC + TC^2 \times TA)(TA \times TB^2 + TB \times TC^2 + TC \times TA^2)$$

W. 68 In all tetrahedron $ABCD$ holds

- $\sum_{cyc} \frac{h_a - r}{h_a + r} \geq \sum_{cyc} \frac{h_a^t - r^t}{(h_a + r)^t}$
- $\sum_{cyc} \frac{2r_a - r}{2r_a + r} \geq \sum_{cyc} \frac{2r_a^t - r^t}{(2r_a + r)^t}$

for all $t \in [0, 1]$

W. 69 Denote $\overline{w_a}, \overline{w_b}, \overline{w_c}$ the external angle-bisectors in triangle ABC, prove that

$$\sum_{cyc} \frac{1}{w_a} \leq \sqrt{\frac{(s^2 - r^2 - 4Rr)(8R^2 - s^2 - r^2 - 2Rr)}{8s^2R^2r}}$$

W. 70 If $x \in \left(0, \frac{\pi}{2}\right)$ then

$$\left(\frac{\sin\left(\frac{\pi}{2}\sin x\right)}{\sin x}\right)^2 + \left(\frac{\sin\left(\frac{\pi}{2}\cos x\right)}{\cos x}\right)^2 \geq 3$$
