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by parmenides51, orl, Amir Hossein

– Day 1

1 Given a right n -angle $A_1A_2\dots A_n$, $n\geq 4$, and a point M inside it. Prove the inequality

$$\sin(\angle A_1MA_2)+\sin(\angle A_2MA_3)+\ldots+\sin(\angle A_nMA_1)>\sin\frac{2\pi}{n}+(n-2)sin\frac{\pi}{n}$$

2 2500 chess kings have to be placed on a 100×100 chessboard so that

(i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);
(ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

Proposed by Sergei Berlov, Russia

3 Given a positive integer $n>2$. Prove that there exists a natural K such that for all integers $k\geq K$ on the open interval $(k^n,\ (k+1)^n)$ there are n different integers, the product of which is the n-th power of an integer.

– Day 2

4 Suppose an ordered set of $(a_1,\ a_2,\ \ldots,\ a_n)$ real numbers, $n\geq 3$. It is possible to replace the number a_i, $i=\overline{2,\ n-1}$ by the number a_i^* that $a_i+a_i^*=a_{i-1}+a_{i+1}$. Let $(b_1,\ b_2,\ \ldots,\ b_n)$ be the set with the largest sum of numbers that can be obtained from this, and $(c_1,\ c_2,\ \ldots,\ c_n)$ is a similar set with the least amount.
For the odd $n\geq 3$ and set $(1,\ 3,\ \ldots,\ n,\ 2,\ 4,\ \ldots,\ n-1)$ find the values of the expressions $b_1+b_2+\ldots+b_n$ and $c_1+c_2+\ldots+c_n$.

5 Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f:\mathbb{Q}^+\mapsto\mathbb{Q}^+$ which satisfy the following equation for all $x,y\in\mathbb{Q}^+:$

$$f\left(f(x)^2y\right)=x^3f(xy).$$

Proposed by Thomas Huber, Switzerland

6 The circle ω inscribed in triangle ABC touches its sides AB, BC, CA at points K, L, M respectively. In the arc KL of the circle ω that does not contain the point M, we select point S. Denote by P, Q, R, T the intersection points of straight AS and KM, ML and SC, LP and KQ, AQ and PC respectively. It turned out that the points R, S and M are collinear. Prove that the point T also lies on the line SM.

– Day 3

7 Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m\left(2^{n+1} - 1\right).$$

Proposed by Angelo Di Pasquale, Australia

8 Is there an increasing sequence of integers $0 = a_0 < a_1 < a_2 < \ldots$ for which the following two conditions are satisfied simultaneously:
1) any natural number can be given as $a_i + a_j$ for some (possibly equal) $i \geq 0$, $j \geq 0$,
2) $a_n > \frac{n^2}{16}$ for all natural n?

9 Inside the inscribed quadrilateral $ABCD$, a point P is marked such that $\angle PBC = \angle PDA$, $\angle PCB = \angle PAD$. Prove that there exists a circle that touches the straight lines AB and CD, as well as the circles circumscribed by the triangles ABP and CDP.

– Day 4

10 Let H be the point of intersection of the altitudes AP and CQ of the acute-angled triangle ABC. The points E and F are marked on the median BM such that $\angle APE = \angle BAC$, $\angle CQF = \angle BCA$, with point E lying inside the triangle APB and point F is inside the triangle CQB. Prove that the lines AE, CF, and BH intersect at one point.

11 Let $P(x)$ and $Q(x)$ be polynomials with real coefficients such that $P(0) > 0$ and all coefficients of the polynomial $S(x) = P(x) \cdot Q(x)$ are integers. Prove that for any positive x the inequality holds:

$$S(x^2) - S^2(x) \leq \frac{1}{4}(P^2(x^3) + Q(x^3)).$$

12 Let n be a natural number. Consider all permutations $(a_1,\ \ldots,\ a_{2n})$ of the first $2n$ natural numbers such that the numbers $|a_{i+1} - a_i|,\ i = 1,\ \ldots,\ 2n-1,$ are pairwise different. Prove that $a_1 - a_{2n} = n$ if and only if $1 \leq a_{2k} \leq n$ for all $k = 1,\ \ldots,\ n$.