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by parmenides51, April

– Day 1

1 There are 2010 red cards and 2010 white cards. All of these 4020 cards are shuffled and dealt in two randomly to each of the 2010 round table players. The game consists of several rounds, each of which players simultaneously hand over cards to each other according to the following rules. If a player holds at least one red card, he passes one red card to the player sitting to his left, otherwise he transfers one white card to the left. The game ends after the round when each player has one red card and one white card. Determine as many rounds as possible.

2 Let $ABCD$ be a quadrilateral inscribled in a circle with the center O, P be the point of intersection of the diagonals AC and BD, $BC \nparallel AD$. Rays AB and DC intersect at the point E. The circle with center I inscribed in the triangle EBC touches BC at point T_1. The E-excircle with center J in the triangle EAD touches the side AD at the point T_2. Line IT_1 and JT_2 intersect at Q. Prove that the points O, P, and Q lie on a straight line.

3 Find all functions f from the set of real numbers into the set of real numbers which satisfy for all x, y the identity

$$f\left(xf(x+y)\right) = f\left(yf(x)\right) + x^2$$

Proposed by Japan

– Day 2

4 For the nonnegative numbers a, b, c prove the inequality:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq \frac{5}{2}$$

5 Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram.
Prove that $GR = GS$.

Proposed by Hossein Karke Abadi, Iran

6 Find all pairs of odd integers a and b for which there exists a natural number$c$ such that the number $\frac{c^n+1}{2^na+b}$ is integer for all natural n.

– Day 3

7 Denote in the triangle ABC by h the length of the height drawn from vertex A, and by $\alpha = \angle BAC$. Prove that the inequality $AB + AC \geq BC \cdot \cos\alpha + 2h \cdot \sin\alpha$. Are there triangles for which this inequality turns into equality?

8 Consider an infinite sequence of positive integers in which each positive integer occurs exactly once. Let $\{a_n\}, n \geq 1$ be such a sequence. We call it *consistent* if, for an arbitrary natural k and every natural n, m such that $a_n < a_m$, the inequality $a_{kn} < a_{km}$ also holds. For example, the sequence $a_n = n$ is consistent .
a) Prove that there are consistent sequences other than $a_n = n$.
b) Are there consistent sequences for which $a_n \neq n, n \geq 2$?
c) Are there consistent sequences for which $an \neq n, n \geq 1$?

9 Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighbouring buckets, empties them to the river and puts them back. Then the next round begins. The Stepmother goal's is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

Proposed by Gerhard Woeginger, Netherlands

– Day 4

10 A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.
(a) Prove that there exist distinct positive integers a and b such that all the number $P(1)$, $P(2)$,. . ., $P(50)$ are balanced.
(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

Proposed by Jorge Tipe, Peru

11 Let ABC be the triangle in which $AB > AC$. Circle ω_a touches the segment of the BC at point D, the extension of the segment AB towards point B at the point F, and the extension of the segment AC towards point C at the point E. The ray AD intersects circle ω_a for second time at point M. Denote the circle circumscribed around the triangle CDM by ω. Circle ω intersects the segment DF at N. Prove that $FN > ND$.

12 Is there a positive integer n for which the following holds:
for an arbitrary rational r there exists an integer b and non-zero integers $a_1, a_2, ..., a_n$ such that

$r = b + \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}$?