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by parmenides51, April, torbich

– Day 1

1 Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, assume it does not lie on line AB. Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

Proposed by Charles Leytem, Luxembourg

2 Let a, b, c are sides of a triangle. Find the least possible value k such that the following inequality always holds: $\left|\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}\right|<k$
(Vitaly Lishunov)

3 Let S be a set consisting of n elements, F a set of subsets of S consisting of 2^{n-1} subsets such that every three such subsets have a non-empty intersection.
a) Show that the intersection of all subsets of F is not empty.
b) If you replace the number of sets from 2^{n-1} with $2^{n-1}-1$, will the previous answer change?

– Day 2

4 Let n be some positive integer. Find all functions $f : R^{+} \to R$ (i.e., functions defined by the set of all positive real numbers with real values) for which equality holds $f\left(x^{n+1}+y^{n+1}\right) = x^{n}f\left(x\right)+y^{n}f\left(y\right)$ for any positive real numbers x, y

5 Let A, B, C, D, E be consecutive points on a circle with center O such that $AC = BD = CE = DO$. Let H_1, H_2, H_3 be the orthocenters triangles ACD, BCD, BCE respectively. Prove that the triangle $H_1H_2H_3$ is right.

6 Find all odd prime numbers p for which there exists a natural number g for which the sets

$$A=\left\{\left(k^2+1\right)\mod p|\,k=1,2,\ldots,\frac{p-1}{2}\right\}$$

and

$$B=\left\{g^k\mod\ p|\,k=1,2,...,\frac{p-1}{2}\right\}$$

are equal.

– Day 3

7 Let a_1, a_2, ..., a_n be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1$, $3a_2$, ..., $3a_n$.

Proposed by Mohsen Jamaali, Iran

8 Two circles γ_1, γ_2 are given, with centers at points O_1, O_2 respectively. Select a point K on circle γ_2 and construct two circles, one γ_3 that touches circle γ_2 at point K and circle γ_1 at a point A, and the other γ_4 that touches circle γ_2 at point K and circle γ_1 at a point B. Prove that, regardless of the choice of point K on circle γ_2, all lines AB pass through a fixed point of the plane.

9 Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair (f, g) of functions from S into S is a *Spanish Couple* on S, if they satisfy the following conditions:
(i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all x, $y \in S$ with $x < y$;

(ii) The inequality $f\left(g\left(g\left(x\right)\right)\right) < g\left(f\left(x\right)\right)$ holds for all $x \in S$.

Decide whether there exists a Spanish Couple - on the set $S = \mathbb{N}$ of positive integers; - on the set $S = \{a - \frac{1}{b} : a, b \in \mathbb{N}\}$

Proposed by Hans Zantema, Netherlands

– Day 4

10 Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

Proposed by John Cuya, Peru

11 Suppose that integers are given $m < n$. Consider a spreadsheet of size $n \times n$, whose cells arbitrarily record all integers from 1 to n^2. Each row of the table is colored in yellow m the largest elements. Similarly, the blue colors the m of the largest elements in each column. Find the smallest number of cells that are colored yellow and blue at a time

12 Denote an acute-angle $\triangle ABC$ with sides a, b, c respectively by H_a, H_b, H_c the feet of altitudes

h_a, h_b, h_c. Prove the inequality:

$$\frac{h_a^2}{a^2 - CH_a^2} + \frac{h_b^2}{b^2 - AH_b^2} + \frac{h_c^2}{c^2 - BH_c^2} \geq 3$$

(Dmitry Petrovsky)