Global Quarantine Mathematical Olympiad 2020

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– Beginner Exam

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1 Find all quadruples of real numbers (a, b, c, d) such that the equalities

$$X^2 + aX + b = (X-a)(X-c) \text{ and } X^2 + cX + d = (X-b)(X-d)$$

hold for all real numbers X.

Morteza Saghafian, Iran

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2 The Bank of Zrich issues coins with an H on one side and a T on the other side. Alice has n of these coins arranged in a line from left to right. She repeatedly performs the following operation: if some coin is showing its H side, Alice chooses a group of consecutive coins (this group must contain at least one coin) and flips all of them; otherwise, all coins show T and Alice stops. For instance, if $n = 3$, Alice may perform the following operations: $THT \to HTH \to HHH \to TTH \to TTT$. She might also choose to perform the operation $THT \to TTT$.

For each initial configuration C, let $m(C)$ be the minimal number of operations that Alice must perform. For example, $m(THT) = 1$ and $m(TTT) = 0$. For every integer $n \geq 1$, determine the largest value of $m(C)$ over all 2^n possible initial configurations C.

Massimiliano Foschi, Italy

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3 Let A and B be two distinct points in the plane. Let M be the midpoint of the segment AB, and let ω be a circle that goes through A and M. Let T be a point on ω such that the line BT is tangent to ω. Let X be a point (other than B) on the line AB such that $TB = TX$, and let Y be the foot of the perpendicular from A onto the line BT.

Prove that the lines AT and XY are parallel.

Navneel Singhal, India

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4 For all real numbers x, we denote by $\lfloor x \rfloor$ the largest integer that does not exceed x. Find all functions f that are defined on the set of all real numbers, take real values, and satisfy the equality

$$f(x+y) = (-1)^{\lfloor y \rfloor} f(x) + (-1)^{\lfloor x \rfloor} f(y)$$

for all real numbers x and y.

Navneel Singhal, India

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5 Let n and k be positive integers such that $k \leq 2^n$. Banana and Corona are playing the following variant of the guessing game. First, Banana secretly picks an integer x such that $1 \leq x \leq n$. Corona will attempt to determine x by asking some questions, which are described as follows. In each turn, Corona chooses k distinct subsets of $\{1, 2, \ldots, n\}$ and, for each chosen set S, asks the question "Is x in the set S?".

Banana picks one of these k questions and tells both the question and its answer to Corona, who can then start another turn.

Find all pairs (n, k) such that, regardless of Banana's actions, Corona could determine x in finitely many turns with absolute certainty.

Pitchayut Saengrungkongka, Thailand

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6 For every integer n not equal to 1 or -1, define $S(n)$ as the smallest integer greater than 1 that divides n. In particular, $S(0) = 2$. We also define $S(1) = S(-1) = 1$.

Let f be a non-constant polynomial with integer coefficients such that $S(f(n)) \leq S(n)$ for every positive integer n. Prove that $f(0) = 0$.

Note: A non-constant polynomial with integer coefficients is a function of the form $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_kx^k$, where k is a positive integer and $a_0, a_1, \ldots, a_k$ are integers such that $a_k \neq 0$.

Pitchayut Saengrungkongka, Thailand

– Advanced Exam

– Day 1

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1 Let ABC be a triangle with incentre I. The incircle of the triangle ABC touches the sides AC and AB at points E and F respectively. Let ℓ_B and ℓ_C be the tangents to the circumcircle of BIC at B and C respectively. Show that there is a circle tangent to EF, ℓ_B and ℓ_C with centre on the line BC.

Proposed by Navneel Singhal, India

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2 Geoff has an infinite stock of sweets, which come in n flavours. He arbitrarily distributes some of the sweets amongst n children (a child can get sweets of any subset of all flavours, including the empty set). Call a distribution $k-$*nice* if every group of k children together has sweets in at least k flavours. Find all subsets S of $\{1, 2, \ldots, n\}$ such that if a distribution of sweets is s-nice for all $s \in S$, then it is s-nice for all $s \in \{1, 2, \ldots, n\}$.

Proposed by Kyle Hess, USA

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3 We call a set of integers *special* if it has 4 elements and can be partitioned into 2 disjoint subsets $\{a, b\}$ and $\{c, d\}$ such that $ab - cd = 1$. For every positive integer n, prove that the set $\{1, 2, \dots, 4n\}$ cannot be partitioned into n disjoint special sets.

Proposed by Mohsen Jamali, Iran

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4 Prove that, for all sufficiently large integers n, there exists n numbers $a_1, a_2, \dots, a_n$ satisfying the following three conditions:

- Each number a_i is equal to either $-1, 0$ or 1.
- At least $\frac{2n}{5}$ of the numbers $a_1, a_2, \dots, a_n$ are non-zero.
- The sum $\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n}$ is 0.

Note: Results with 2/5 replaced by a constant c will be awarded points depending on the value of c

Proposed by Navneel Singhal, India; Kyle Hess, USA; and Vincent Jug, France

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– Day 2

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5 Let $\mathbb{Q}$ denote the set of rational numbers. Determine all functions $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ such that, for all $x, y \in \mathbb{Q}$,

$$f(x)f(y+1) = f(xf(y)) + f(x)$$

Nicols Lpez Funes and Jos Luis Narbona Valiente, Spain

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6 Decide whether there exist infinitely many triples (a, b, c) of positive integers such that all prime factors of $a! + b! + c!$ are smaller than 2020.

Pitchayut Saengrungkongka, Thailand

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7 Each integer in $\{1, 2, 3, ..., 2020\}$ is coloured in such a way that, for all positive integers a and b such that $a + b \leq 2020$, the numbers a, b and $a + b$ are not coloured with three different colours. Determine the maximum number of colours that can be used.

Massimiliano Foschi, Italy

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8 Let ABC be an acute scalene triangle, with the feet of A, B, C onto BC, CA, AB being D, E, F respectively. Let W be a point inside ABC whose reflections over BC, CA, AB are W_a, W_b, W_c respectively. Finally, let N and I be the circumcenter and the incenter of $W_aW_bW_c$ respectively. Prove that, if N coincides with the nine-point center of DEF, the line WI is parallel to the Euler line of ABC.

Proposed by Navneel Singhal, India and Massimiliano Foschi, Italy

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