Switzerland Team Selection Test 2019

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– Day 1

1 Let ABC be a triangle and D, E, F be the foots of altitudes drawn from A, B, C respectively. Let H be the orthocenter of ABC. Lines EF and AD intersect at G. Let K the point on circumcircle of ABC such that AK is a diameter of this circle. AK cuts BC in M. Prove that GM and HK are parallel.

2 Find the largest prime p such that there exist positive integers a, b satisfying

$$p = \frac{b}{2}\sqrt{\frac{a-b}{a+b}}.$$

3 Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x\in F} 1/x = \sum_{x\in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

– Day 2

4 Let p be a prime number. Find all polynomials P with integer coefficients with the following properties: (a) $P(x) > x$ for all positive integers x. (b) The sequence defined by $p_0 := p$, $p_{n+1} := P(p_n)$ for all positive integers n, satisfies the property that for all positive integers m there exists some $l \geq 0$ such that $m \mid p_l$.

5 Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

6 Let (a, b) be a pair of natural numbers. Henning and Paul play the following game. At the beginning there are two piles of a and b coins respectively. We say that (a, b) is the *starting position* of the game. Henning and Paul play with the following rules: • They take turns alternatively where Henning begins. • In every step each player either takes a positive integer number of coins from one of the two piles or takes same natural number of coins from both piles. • The

player how take the last coin wins.
Let A be the set of all positive integers like a for which there exists a positive integer $b < a$ such that Paul has a wining strategy for the starting position (a, b). Order the elements of A to construct a sequence $a_1 < a_2 < a_3 < \dots$ (a) Prove that A has infinity many elements. (b) Prove that the sequence defined by $m_k := a_{k+1} - a_k$ will never become periodic. (This means the sequence m_{k_0+k} will not be periodic for any choice of k_0)

– Day 3

7 Prove that for all positive integers n there are positive integers a, b such that

$$n \mid 4a^2 + 9b^2 - 1.$$

8 Let k, n, r be positive integers and $r < n$. Quirin owns $kn + r$ black and $kn + r$ white socks. He want to clean his cloths closet such there does not exist $2n$ consecutive socks n of which black and the other n white. Prove that
he can clean his closet in the desired manner if and only if $r \geq k$ and $n > k + r$.

9 Let ABC be an acute triangle with $AB < AC$. E, F are foots of the altitudes drawn from B, C respectively. Let M be the midpoint of segment BC. The tangent at A to the circumcircle of ABC cuts BC in P and EF cuts the parallel to BC from A at Q. Prove that PQ is perpendicular to AM.

– Day 4

10 Let $n \geq 5$ be an integer. A shop sells balls in n different colors. Each of $n + 1$ children bought three balls with different colors, but no two children bought exactly the same color combination. Show that there are at least two children who bought exactly one ball of the same color.

11 Let n be a positive integer. Determine whether there exists a positive real number $\epsilon > 0$ (depending on n) such that for all positive real numbers $x_1, x_2, \dots, x_n$, the inequality

$$\sqrt[n]{x_1 x_2 \dots x_n} \leq (1 - \epsilon)\frac{x_1 + x_2 + \dots + x_n}{n} + \epsilon \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}},$$

holds.

12 Define the sequence $a_0, a_1, a_2, \dots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.