

Tournament Of Towns year 2020

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– Spring 2020

– Junior O Level

- 1** The Quadrumland map is a 6×6 square where each square cell is either a kingdom or a disputed territory. There are 27 kingdoms and 9 disputed territories. Each disputed territory is claimed by those and only those kingdoms that are neighbouring with it (adjacent by an edge or a vertex). Is it possible that for each disputed territory the numbers of claims are different?

You can discuss your solutions here

- 2** *What is the maximum number of distinct integers in a row such that the sum of any 11 consequent integer*
I'm posting this problem for people to discuss
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- 3** Let $ABCD$ be a rhombus, let $APQC$ be a parallelogram such that the point B lies inside it and the side AP is equal to the side of the rhombus. Prove that B is the orthocenter of the triangle DPQ .

Egor Bakaev

- 4** For some integer n the equation $x^2 + y^2 + z^2 - xy - yz - zx = n$ has an integer solution x, y, z . Prove that the equation $x^2 + y^2 - xy = n$ also has an integer solution x, y .

Alexandr Yuran

- 5** On the 8×8 chessboard there are two identical markers in the squares $a1$ and $c3$. Alice and Bob in turn make the following moves (the first move is Alice's): a player picks any marker and moves it horizontally to the right or vertically upwards through any number of squares. The aim of each player is to get to the square $h8$. Which player has a winning strategy no matter what does his opponent? (There may be only one marker on a square, the markers may not go through each other.)

The 8×8 chessboard consists of columns lettered a to h from left to right and rows numbered $1-8$ from bottom to top

<https://cdn.artofproblemsolving.com/attachments/1/f/4c5548f606fda915e0a50a8cf886ff93e1f80.png>

– Junior A Level

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- 1 Does there exist a positive integer that is divisible by 2020 and has equal numbers of digits 0, 1, 2, ..., 9?

Mikhail Evdokimov

- 2 Three legendary knights are fighting against a multiheaded dragon. Whenever the first knight attacks, he cuts off half of the current number of heads plus one more. Whenever the second knight attacks, he cuts off one third of the current number of heads plus two more. Whenever the third knight attacks, he cuts off one fourth of the current number of heads plus three more. They repeatedly attack in an arbitrary order so that at each step an integer number of heads is being cut off. If all the knights cannot attack as the number of heads would become non-integer, the dragon eats them. Will the knights be able to cut off all the dragon's heads if it has $41!$ heads?

Alexey Zaslavsky

- 3 Is it possible to inscribe an N -gon in a circle so that all the lengths of its sides are different and all its angles (in degrees) are integer, where

- a) $N = 19$,
b) $N = 20$?

Mikhail Malkin

- 4 For which integers N it is possible to write real numbers into the cells of a square of size $N \times N$ so that among the sums of each pair of adjacent cells there are all integers from 1 to $2(N-1)N$ (each integer once)?

Maxim Didin

- 5 Let $ABCD$ be an inscribed trapezoid. The base AB is 3 times longer than CD . Tangents to the circumscribed circle at the points A and C intersect at the point K . Prove that the angle KDA is a right angle.

Alexandr Yuran

- 6 Alice has a deck of 36 cards, 4 suits of 9 cards each. She picks any 18 cards and gives the rest to Bob. Now each turn Alice picks any of her cards and lays it face-up onto the table, then Bob similarly picks any of his cards and lays it face-up onto the table. If this pair of cards has the same suit or the same value, Bob gains a point. What is the maximum number of points he can guarantee regardless of Alice's actions?

Mikhail Evdokimov

- 7 Gleb picked positive integers N and a ($a < N$). He wrote the number a on a blackboard. Then each turn he did the following: he took the last number on the blackboard, divided the number N

by this last number with remainder and wrote the remainder onto the board. When he wrote the number 0 onto the board, he stopped. Could he pick N and a such that the sum of the numbers on the blackboard would become greater than $100N$?

Ivan Mitrofanov

– Senior O Level

- 1 Is it possible to fill a 40×41 table with integers so that each integer equals the number of adjacent (by an edge) cells with the same integer?

Alexandr Gribalko

- 2 Alice asserts that after her recent visit to Addis-Ababa she now has spent the New Year inside every possible hemisphere of Earth except one. What is the minimal number of places where Alice has spent the New Year?

Note: we consider places of spending the New Year to be points on the sphere. A point on the border of a hemisphere does not lie inside the hemisphere.

Ilya Dumansky, Roman Krutovsky

- 3 There are 41 letters on a circle, each letter is A or B . It is allowed to replace ABA by B and conversely, as well as to replace BAB by A and conversely. Is it necessarily true that it is possible to obtain a circle containing a single letter repeating these operations?

Maxim Didin

- 4 We say that a nonconstant polynomial $p(x)$ with real coefficients is split into two squares if it is represented as $a(x)+b(x)$ where $a(x)$ and $b(x)$ are squares of polynomials with real coefficients. Is there such a polynomial $p(x)$ that it may be split into two squares:

- a) in exactly one way;
b) in exactly two ways?

Note: two splittings that differ only in the order of summands are considered to be the same.

Sergey Markelov

- 5 Given are two circles which intersect at points P and Q . Consider an arbitrary line ℓ through Q , let the second points of intersection of this line with the circles be A and B respectively. Let C be the point of intersection of the tangents to the circles in those points. Let D be the intersection of the line AB and the bisector of the angle CPQ . Prove that all possible D for any choice of ℓ lie on a single circle.

Alexey Zaslavsky

– Senior A Level

- 1 Consider two parabolas $y = x^2$ and $y = x^2 - 1$. Let U be the set of points between the parabolas (including the points on the parabolas themselves). Does U contain a line segment of length greater than 10^6 ?

Alexey Tolpygo

- 2 Alice had picked positive integers a, b, c and then tried to find positive integers x, y, z such that $a = \text{lcm}(x, y)$, $b = \text{lcm}(x, z)$, $c = \text{lcm}(y, z)$. It so happened that such x, y, z existed and were unique. Alice told this fact to Bob and also told him the numbers a and b . Prove that Bob can find c . (Note: lcm = least common multiple.)

Boris Frenkin

- 3 Is it possible that two cross-sections of a tetrahedron by two different cutting planes are two squares, one with a side of length no greater than 1 and another with a side of length at least 100?

Mikhail Evdokimov

- 4 Henry invited $2N$ guests to his birthday party. He has N white hats and N black hats. He wants to place hats on his guests and split his guests into one or several dancing circles so that in each circle there would be at least two people and the colors of hats of any two neighbours would be different. Prove that Henry can do this in exactly $(2N)!$ different ways. (All the hats with the same color are identical, all the guests are obviously distinct, $(2N)! = 1 \cdot 2 \cdot \dots \cdot (2N)$.)

Gleb Pogudin

- 5 Let $ABCD$ be an inscribed quadrilateral. Let the circles with diameters AB and CD intersect at two points X_1 and Y_1 , the circles with diameters BC and AD intersect at two points X_2 and Y_2 , the circles with diameters AC and BD intersect at two points X_3 and Y_3 . Prove that the lines X_1Y_1 , X_2Y_2 and X_3Y_3 are concurrent.

Maxim Didin

- 6 There are $2n$ consecutive integers on a board. It is permitted to split them into pairs and simultaneously replace each pair by their difference (not necessarily positive) and their sum. Prove that it is impossible to obtain any $2n$ consecutive integers again.

Alexandr Gribalko

- 7 Consider an infinite white plane divided into square cells. For which k it is possible to paint a positive finite number of cells black so that on each horizontal, vertical and diagonal line of cells there is either exactly k black cells or none at all?

A. Dinev, K. Garov, N Belukhov

– Oral (15 March 2020)

1 2020 positive integers are written in one line. Each of them starting with the third is divisible by previous and by the sum of two previous numbers. What is the smallest value the last number can take?

A. Gribalko

2 At heights AA_0, BB_0, CC_0 of an acute-angled non-equilateral triangle ABC , points A_1, B_1, C_1 were marked, respectively, so that $AA_1 = BB_1 = CC_1 = R$, where R is the radius of the circumscribed circle of triangle ABC . Prove that the center of the circumscribed circle of the triangle $A_1B_1C_1$ coincides with the center of the inscribed circle of triangle ABC .

E. Bakaev

3 40 cells were marked on an infinite chessboard. Is it always possible to find a rectangle that contains 20 marked cells?

M. Evdokimov

4 For an infinite sequence a_1, a_2, \dots denote as it's *first derivative* is the sequence $a'_n = a_{n+1} - a_n$ (where $n = 1, 2, \dots$), and her k -th derivative as the first derivative of its $(k - 1)$ -th derivative ($k = 2, 3, \dots$). We call a sequence *good* if it and all its derivatives consist of positive numbers. Prove that if a_1, a_2, \dots and b_1, b_2, \dots are good sequences, then sequence $a_1 \cdot b_1, a_2 \cdot b_2, \dots$ is also a good one.

R. Salimov

5 A triangle is given on a sphere of radius 1, the sides of which are arcs of three different circles of radius 1 centered in the center of a sphere having less than π in length and an area equal to a quarter of the area of the sphere. Prove that four copies of such a triangle can cover the entire sphere.

A. Zaslavsky

6 Given an endless supply of white, blue and red cubes. In a circle arrange any N of them. The robot, standing in any place of the circle, goes clockwise and, until one cube remains, constantly repeats this operation: destroys the two closest cubes in front of him and puts a new one behind him a cube of the same color if the destroyed ones are the same, and the third color if the destroyed two are different colors.

We will call the arrangement of the cubes *good* if the color of the cube remaining at the very end does not depends on where the robot started. We call N *successful* if for any choice of N cubes all their arrangements are good. Find all successful N .

I. Bogdanov
