239 Open Mathematical Olympiad 2015

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–	Grade 10-11
1	Let the incircle of triangle ABC touches the sides AB, BC, CA in C_1, A_1, B_1 respectively. If A_1C_1 cuts the parallel to BC from A at K prove that $\angle KB_1A_1 = 90$.
2	Prove that $\binom{n+k}{n}$ can be written as product of n pairwise coprime numbers $a_1, a_2, \ldots, a_n$ such that $k+i$ is divisible by a_i for all indices i.
3	The edges of a graph G are coloured in two colours. Such that for each colour all the connected components of this graph formed by edges of this colour contains at most $n > 1$ vertices. Prove there exists a proper colouring for the vertices of this graph with n colours.
4	A natural number n is given. Let $f(x, y)$ be a polynomial of degree less than n such that for any positive integers $x, y \leq n, x+y \leq n+1$ the equality $f(x, y) = \frac{x}{y}$ holds. Find $f(0, 0)$.
5	The nodes of a three dimensional unit cube lattice with all three coordinates even are coloured red and blue otherwise. A convex polyhedron with all vertices red is given. Assuming the number of red points on its border is n. How many blue vertices can be on its border?
6	Positive real numbers a, b, c satisfy $$2a^3b + 2b^3c + 2c^3a = a^2b^2 + b^2c^2 + c^2a^2.$$ Prove that $$2ab(a-b)^2 + 2bc(b-c)^2 + 2ca(c-a)^2 \geq (ab+bc+ca)^2.$$
7	Two magicians are about to show the next trick. A circle is drawn on the board with one semicircle marked. Viewers mark 100 points on this circle, then the first magician erases one of them. After this, the second one for the first time looks at the drawing and determines from the remaining 99 points whether the erased point was lying on the marked semicircle. Prove that such a trick will not always succeed.
8	On a circle 100 points are chosen and for each point we wrote the multiple of its distances to the rest. Could the written numbers be $1, 2, \ldots, 100$ in some order?
–	Grade 8-9

1 There are 10 stones of different weights with distinct pairwise sums. We have a special two-tiered balance scale such that only two stones can be put on each cup and then we understand which cup is heavier. Prove that having this scale you can either find the heaviest or the lightest stone.

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2 Same as grade 10-11, 1

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3 Positive integers are colored either blue or red such that if a, b have the same color and $a - 10b$ is a positive integer then $a - 10b, a$ have the same color as well. How many such coloring exist?

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4 On a circle 4 points are chosen and for each point we wrote the multiple of its distances to the rest. Could the written numbers be $1, 2, 3, 4$ in some order?

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5 Edges of a complete graph with $2m$ vertices are properly colored with $2m - 1$ colors. It turned out that for any two colors all the edges colored in one of these two colors can be described as union of several 4-cycles. Prove that m is a power of 2.

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6 The numbers $1, 2, 3, \ldots, 1000$ are written on the board. Patya and Vassya are playing a game. They take turn alternatively erasing a number from the board. Patya begins. If after a turn all numbers (maybe one) on the board be divisible by a natural number greater than 1 the player who last played loses. If after some number of steps the only remaining number on the board be 1 then they call it a draw. Determine the result of the game if they both play their best.

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7 There is a closed polyline with n edges on the plane. We build a new polyline which edges connect the midpoints of two adjacent edges of the previous polyline. Then we erase previous polyline and start over and over. Also we know that each polyline satisfy that all vertices are different and not all of them are collinear. For which n we can get a polyline that is a onvex polygon?

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8 Same as grade 10-11, 6
