239 Open Mathematical Olympiad 2000

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–	Grade 10-11
1	Given pairwise coprime natural numbers x, y, z, t such that $xy + yz + zt = xt$. Prove that the sum of the squares of some two of these numbers is twice the sum of the squares of the two remaining.
2	100 volleyball teams played a one-round tournament. No two matches held at the same time. It turned out that before each match the teams playing against each other had the same number of wins. Find all possible number of wins for the winner of this tournament.
3	For all positive real numbers $a_1, a_2, \dots, a_n$, prove that $$\frac{a_1+a_2}{2}\cdot\frac{a_2+a_3}{2}\cdot\dots\cdot\frac{a_n+a_1}{2}\le\frac{a_1+a_2+a_3}{2\sqrt{2}}\cdot\frac{a_2+a_3+a_4}{2\sqrt{2}}\cdot\dots\cdot\frac{a_n+a_1+a_2}{2\sqrt{2}}.$$
4	A graph is called 2-connected if after removing any vertex the remaining graph is still connected. Prove that for any 2-connected graph with degrees more than two, one can remove a vertex so that the remaining graph is still 2-connected.
5	Let m be a positive integer. Prove that there exist infinitely many prime numbers p such that $m + p^3$ is composite.
6	n cockroaches are sitting on the plane at the vertices of the regular n -gon. They simultaneously begin to move at a speed of v on the sides of the polygon in the direction of the clockwise adjacent cockroach, then they continue moving in the initial direction with the initial speed. Vasya a entomologist moves on a straight line in the plane at a speed of u. After some time, it turned out that Vasya has crushed three cockroaches. Prove that $v = u$.
7	The perpendicular bisectors of the sides AB and BC of a triangle ABC meet the lines BC and AB at the points X and Z, respectively. The angle bisectors of the angles XAC and ZCA intersect at a point B'. Similarly, define two points C' and A'. Prove that the points A', B', C' lie on one line through the incenter I of triangle ABC.
8	Given a set of 102 elements. Is it possible to choose 102 17-element subsets so that the intersection of any two subsets contains no more than 3 elements?
–	Grade 8-9

1 On an infinite checkered plane 100 chips in form of a 10×10 square are given. These chips are rearranged such that any two adjacent (by side) chips are again adjacent, moreover no two chips are in the same cell. Prove that the chips are again in form of a square.

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2 Same as grade 10-11, 1

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3 Let AA_1 and CC_1 be the altitudes of the acute-angled triangle ABC. A line passing through the centers of the inscribed circles the triangles AA_1C and CC_1A intersect the sides of AB and BC triangle ABC at points X and Y. Prove that $BX = BY$.

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4 Is there a 30-digit number such that any number formed by its five consecutive digits is divisible by 13?

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5 Same as grade 10-11, 2

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6 Let $ABCD$ be a convex quadrilateral, and let M and N be the midpoints of its sides AD and BC, respectively. Assume that the points A, B, M, N are concyclic, and the circumcircle of triangle BMC touches the line AB. Show that the circumcircle of triangle AND touches the line AB.

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7 Same as grade 10-11, 3

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8 Same as grade 10-11, 4
