239 Open Mathematical Olympiad 2001

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– Grade 10-11

1 Find all triples of natural numbers a, b, c such that

$$\gcd(a^2, b^2) + \gcd(a, bc) + \gcd(b, ac) + \gcd(c, ab) = 239^2 = ab + c.$$

2 For any positive numbers $a_1, a_2, \dots, a_n$ prove the inequality

$$\left(1+\frac{1}{a_1(1+a_1)}\right)\left(1+\frac{1}{a_2(1+a_2)}\right)\dots\left(1+\frac{1}{a_n(1+a_n)}\right) \geq \left(1+\frac{1}{p(1+p)}\right)^n,$$

where $p = \sqrt[n]{a_1a_2\dots a_n}$.

3 The circles S_1 and S_2 intersect at points A and B. Circle S_3 externally touches S_1 and S_2 at points C and D respectively. Let PQ be a chord cut by the line AB on circle S_3, and K be the midpoint of CD. Prove that $\angle PKC = \angle QKC$.

4 Integers are placed on every cell of an infinite checkerboard. For each cell if it contains integer a then the sum of
the numbers in the cell under it and the cell right to it is $2a + 1$. Prove that in every infinite diagonal row of direction *top-right down-left* all numbers are different.

5 Let $P(x)$ be a monic polynomial with integer coefficients of degree 10. Prove that there exist distinct positive integers a, b not exceeding 101 such that $P(a) - P(b)$ is divisible by 101.

6 On the plane 1000 lines are drawn, among which there are no parallel lines. From any seven of these lines, some three pass through one point. But no more than 500 lines pass through each point. Prove that there are three points such that each line contains at least of of them.

7 The quadrangle $ABCD$ contains two circles of radii R_1 and R_2 tangent externally. The first circle touches the sides of DA,AB and BC, moreover, the sides of AB at the point E. The second circle touches sides BC, CD and DA, and sides CD at F. Diagonals of the quadrangle intersect at O. Prove that $OE + OF \leq 2(R_1 + R_2)$.

(F. Bakharev, S. Berlov)

8 Assume that the connected graph G has n vertices all with degree at least three. Prove that there exists a spanning tree of G with more than $\frac{2}{9}n$ leaves.

– Grade 8-9

1 A square $n \times n$, $(n > 2)$ contains nonzero real numbers. It is known that every number is exactly k times smaller than the sum of all the numbers in its row or sum of all number in its column. For which real numbers k is this possible?

2 In a convex quadrangle $ABCD$, the rays DA and CB intersect at point Q, and the rays BA and CD at the point P. It turned out that $\angle AQB = \angle APD$. The bisectors of the angles $\angle AQB$ and $\angle APD$ intersect the sides quadrangle at points X, Y and Z, T respectively. Circumscribed circles of triangles ZQT and XPY intersect at K inside quadrangle. Prove that K lies on the diagonal AC.

3 The numbers $1, 2, \ldots, 1999$ are written on the board. Two players take turn choosing a, b from the board and erasing them then writing one of ab, $a + b$, $a - b$. The first player wants the last number on the board to be divisible by 1999, the second player want to stop him. Determine the winner.

4 Same as grade 10-11, 1

5 The circles S_1 and S_2 intersect at points A and B. Circle S_3 externally touches S_1 and S_2 at points C and D respectively. Let K be the midpoint of the chord cut by the line AB on circles S_3. Prove that $\angle CKA = \angle DKA$.

6 On the plane 100 lines are drawn, among which there are no parallel lines. From any five of these lines, some three pass through one point. Prove that there are two points such that each line contains at least of of them.

7 Same as grade 10-11, 2

8 In a graph with $2n - 1$ vertices throwing out any vertex the remaining graph has a complete subgraph with n vertices. Prove that the initial graph has a complete subgraph with $n + 1$ vertices.