**Romania Team Selection Tests 2018**

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– Day 1

**1** Find the least number $c$ satisfyng the condition $\sum_{i=1}^n {x_i}^2 \leq cn$
and all real numbers $x_1, x_2, ..., x_n$ are greater than or equal to $-1$ such that $\sum_{i=1}^n {x_i}^3 = 0$

**2** Let $ABC$ be a triangle, let $I$ be its incenter, let $\Omega$ be its circumcircle, and let $\omega$ be the $A$- mixtilinear incircle. Let $D, E$ and $T$ be the intersections of $\omega$ and $AB, AC$ and $\Omega$, respectively, let the line $IT$ cross $\omega$ again at $P$, and let lines $PD$ and $PE$ cross the line $BC$ at $M$ and $N$ respectively. Prove that points $D, E, M, N$ are concyclic. What is the center of this circle?

**3** Divide the plane into $1$x$1$ squares formed by the lattice points. Let$S$ be the set-theoretic union of a finite number of such cells, and let $a$ be a positive real number less than or equal to 1/4.Show that S can be covered by a finite number of squares satisfying the following three conditions:
1) Each square in the cover is an array of $1$x$1$ cells
2) The squares in the cover have pairwise disjoint interios and
3)For each square $Q$ in the cover the ratio of the area $S \cap Q$ to the area of Q is at least $a$ and at most $a(\lfloor a^{-1/2} \rfloor)^2$

**4** Given an non-negative integer $k$, show that there are infinitely many positive integers $n$ such that the product of any $n$ consecutive integers is divisible by $(n+k)^2+1$.

– Day 2

**1** Let $ABC$ be a triangle, and let $M$ be a point on the side $(AC)$ .The line through $M$ and parallel to $BC$ crosses $AB$ at $N$. Segments $BM$ and $CN$ cross at $P$, and the circles $BNP$ and $CMP$ cross again at $Q$. Show that angles $BAP$ and $CAQ$ are equal.

**2** Show that a number $n(n+1)$ where $n$ is positive integer is the sum of 2 numbers $k(k+1)$ and $m(m+1)$ where $m$ and $k$ are positive integers if and only if the number $2n^2+2n+1$ is composite.

**3** Consider a 4-point configuration in the plane such that every 3 points can be covered by a strip of a unit width. Prove that:
1) the four points can be covered by a strip of length at most $\sqrt{2}$ and
2)if no strip of length less that $\sqrt{2}$ covers all the four points, then the points are vertices of a square of length $\sqrt{2}$

**4** Let $D$ be a non-empty subset of positive integers and let $d$ be the greatest common divisor of $D$, and let $d\mathbb{Z} = [dn : n \in \mathbb{Z}]$. Prove that there exists a bijection $f : \mathbb{Z} \to d\mathbb{Z}$ such that $|f(n+1) - f(n)|$ is member of $D$ for every integer $n$.

**–** Day 3

**1** Let $ABCD$ be a cyclic quadrilateral and let its diagonals $AC$ and $BD$ cross at $X$. Let $I$ be the incenter of $XBC$, and let $J$ be the center of the circle tangent to the side $BC$ and the extensions of sides $AB$ and $DC$ beyond $B$ and $C$. Prove that the line $IJ$ bisects the arc $BC$ of circle $ABCD$, not containing the vertices $A$ and $D$ of the quadrilateral.

**2** Given a square-free integer $n > 2$, evaluate the sum $\sum_{k=1}^{(n-2)(n-1)} \lfloor (kn)^{1/3} \rfloor$.

**3**

For every integer $n \geq 2$ let $B_n$ denote the set of all binary $n$-nuples of zeroes and ones, and split $B_n$ into equivalence classes by letting two $n$-nuples be equivalent if one is obtained from the another by a cyclic permutation.(for example 110, 011 and 101 are equivalent). Determine the integers $n \geq 2$ for which $B_n$ splits into an odd number of equivalence classes.

**4** Given two positives integers $m$ and $n$, prove that there exists a positive integer $k$ and a set $S$ of at least $m$ multiples of $n$ such that the numbers $\frac{2^k \sigma(s)}{s}$ are odd for every $s \in S$. $\sigma(s)$ is the sum of all positive integers of $s$ (1 and $s$ included).

**–** Day 4

**1** Let $O$ be the circumcenter of an acute triangle $ABC$. Line $OA$ intersects the altitudes of $ABC$ through $B$ and $C$ at $P$ and $Q$, respectively. The altitudes meet at $H$. Prove that the circumcenter of triangle $PQH$ lies on a median of triangle $ABC$.

**2** Let $n$ be a positive integer. Define a chameleon to be any sequence of $3n$ letters, with exactly $n$ occurrences of each of the letters $a, b,$ and $c$. Define a swap to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon $X$ , there exists a chameleon $Y$ such that $X$ cannot be changed to $Y$ using fewer than $3n^2/2$ swaps.

**3** Given an integer $n \geq 2$ determine the integral part of the number $\sum_{k=1}^{n-1} \frac{1}{(1+\frac{1}{n})\dots(1+\frac{k}{n})}$ - $\sum_{k=1}^{n-1}(1-\frac{1}{n})\dots(1-\frac{k}{n})$

**–** Day 5

**1** In triangle $ABC$, let $\omega$ be the excircle opposite to $A$. Let $D, E$ and $F$ be the points where $\omega$ is tangent to $BC, CA$, and $AB$, respectively. The circle $AEF$ intersects line $BC$ at $P$ and $Q$. Let

$M$ be the midpoint of $AD$. Prove that the circle $MPQ$ is tangent to $\omega$.

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**2** Determine all integers $n \geq 2$ having the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by $n$, there exists an index $1 \leq i \leq n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \ldots + a_{i+n-1}$$

is divisible by $n$. Here, we let $a_i = a_{i-n}$ when $i > n$.

*Proposed by Warut Suksompong, Thailand*

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**3** Let $n > 1$ be a given integer. An $n \times n \times n$ cube is composed of $n^3$ unit cubes. Each unit cube is painted with one colour. For each $n \times n \times 1$ box consisting of $n^2$ unit cubes (in any of the three possible orientations), we consider the set of colours present in that box (each colour is listed only once). This way, we get $3n$ sets of colours, split into three groups according to the orientation.

It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of $n$, the maximal possible number of colours that are present.

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