239 Open Mathematical Olympiad 2017

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–	Grade 10-11
1	On the side AC of triangle ABC point D is chosen. Let I_1, I_2, I be the incenters of triangles ABD, BCD, ABC respectively. It turned out that I is the orthocentre of triangle I_1I_2B. Prove that BD is an altitude of triangle ABC.
2	Find all composite numbers n such that for each decomposition of $n = xy$, $x + y$ be a power of 2.
3	Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real number x, y, $$(y + 1)f(yf(x)) = yf(x(y + 1)).$$
4	A polynomial $f(x)$ with integer coefficients is given. We define $d(a, k) = \|f^k(a) - a\|$. It is known that for each integer a and natural number k, $d(a, k)$ is positive. Prove that for all such a, k, $$d(a, k) \geq \frac{k}{3}.$$ ($f^k(x) = f(f^{k-1}(x)), f^0(x) = x$.)
5	A school has three classes. Some pairs of children from different classes are enemies (there are no enemies in a class). It is known that every child from the first class has as many enemies in the second class as in the third; the same is true for other classes. Prove that the number of pairs of children from classes having a common enemy is not less than the number of pairs of children being enemies.
7	An invisible tank is on a 100×100 table. A cannon can fire at any k cells of the board after that the tank will move to one of the adjacent cells (by side). Then the progress is repeated. Find the smallest value of k such that the cannon can definitely shoot the tank after some time.
6	Given a circumscribed quadrilateral $ABCD$ in which $$\sqrt{2}(BC - BA) = AC.$$ Let X be the midpoint of AC and Y a point on the angle bisector of B such that XD is the angle bisector of BXY. Prove that BD is tangent to the circumcircle of DXY.

8 Assume that the connected graph G has n vertices all with degree at least three. Prove that there exists a spanning tree of G with more than $\frac{2}{9}n$ leaves.

– Grade 8-9

1 Denote every permutation of $1, 2, \ldots, n$ as $\sigma = (a_1, a_2, \ldots, n)$. Prove that the sum

$$\sum \frac{1}{(a_1)(a_1 + a_2)(a_1 + a_2 + a_3) \ldots (a_1 + a_2 + \cdots + a_n)}$$

taken over all possible permutations σ equals $\frac{1}{n!}$.

2 Inside the circle ω through points A, B point C is chosen. An arbitrary point X is selected on the segment BC. The ray AX cuts the circle in Y. Prove that all circles CXY pass through a two fixed points that is they intersect and are coaxial, independent of the position of X.

3 Same as grade 10-11, 2

4 An invisible tank is on a 100×100 table. A cannon can fire at any 60 cells of the board after that the tank will move to one of the adjacent cells (by side). Then the progress is repeated. Can the cannon grantee to shoot the tank?

5 Given a quadrilateral $ABCD$ in which

$$\sqrt{2}(BC - BA) = AC.$$

Let X be the midpoint of AC. Prove that $2\angle BXD = \angle DAB - \angle DCB$.

6 The natural numbers $y > x$ are written on the board. Vassya decides to write the reminder of one number on the board to some other non-zero number in each step. Prove that Vassya can find a natural number k such that if $y > k$ then the number distinct numbers on the board after arbitrary number of steps does not exceed $\frac{y}{1000000}$.

7 Find the greatest possible value of $s > 0$, such that for any positive real numbers a, b, c,

$$\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right)^2 \geq s\left(\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab}\right).$$

8 Same as grade 10-11, 5