APMO 2020

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1 Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

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2 Show that $r = 2$ is the largest real number r which satisfies the following condition:

If a sequence $a_1, a_2, \ldots$ of positive integers fulfills the inequalities

$$a_n \leq a_{n+2} \leq \sqrt{a_n^2 + ra_{n+1}}$$

for every positive integer n, then there exists a positive integer M such that $a_{n+2} = a_n$ for every $n \geq M$.

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3 Determine all positive integers k for which there exist a positive integer m and a set S of positive integers such that any integer $n > m$ can be written as a sum of distinct elements of S in exactly k ways.

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4 Let $\mathbb{Z}$ denote the set of all integers. Find all polynomials $P(x)$ with integer coefficients that satisfy the following property:

For any infinite sequence $a_1, a_2, \ldots$ of integers in which each integer in $\mathbb{Z}$ appears exactly once, there exist indices $i < j$ and an integer k such that $a_i + a_{i+1} + \cdots + a_j = P(k)$.

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5 Let $n \geq 3$ be a fixed integer. The number 1 is written n times on a blackboard. Below the blackboard, there are two buckets that are initially empty. A move consists of erasing two of the numbers a and b, replacing them with the numbers 1 and $a + b$, then adding one stone to the first bucket and $\gcd(a, b)$ stones to the second bucket. After some finite number of moves, there are s stones in the first bucket and t stones in the second bucket, where s and t are positive integers. Find all possible values of the ratio $\frac{t}{s}$.
