Belarusian National Olympiad 2001

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– Day 1

1 On the Cartesian coordinate plane, the graph of the parabola $y = x^2$ is drawn. Three distinct points A, B, and C are marked on the graph with A lying between B and C. Point N is marked on BC so that AN is parallel to the y-axis. Let K_1 and K_2 are the areas of triangles ABN and ACN, respectively. Express AN in terms of K_1 and K_2.

2 Prove for postitive a and natural n

$$a^n + \frac{1}{a^n} - 2 \geq n^2(a + \frac{1}{a} - 2)$$

3 Three distinct points A, B, and N are marked on the line l, with B lying between A and N. For an arbitrary angle $\alpha \in (0, \frac{\pi}{2})$, points C and D are marked in the plane on the same side of l such that N, C, and D are collinear; $\angle NAD = \angle NBC = \alpha$; and A, B, C, and D are concyclic. Find the locus of the intersection points of the diagonals of $ABCD$ as α varies between 0 and $\frac{\pi}{2}$.

4 The problem committee of a mathematical olympiad prepares some variants of the contest. Each variant contains 4 problems, chosen from a shortlist of n problems, and any two variants have at most one problem in common.
(a) If $n = 14$, determine the largest possible number of variants the problem committee can prepare.
(b) Find the smallest value of n such that it is possible to prepare ten variants of the contest.

– Day 2

5 In the increasing sequence of positive integers $a_1, a_2, \ldots$, the number a_k is said to be funny if it can be represented as the sum of some other terms (not necessarily distinct) of the sequence.
(a) Prove that all but finitely terms of the sequence are funny.
(b) Does the result in (a) always hold if the terms of the sequence can be any positive rational numbers?

6 Let n be a positive integer. Each square of a $(2n-1) \times (2n-1)$ square board contains an arrow, either pointing up, down,to the left, or to the right. A beetle sits in one of the cells. Each year it creeps from one square in the direction of the arrow in that square, either reaching another square or leaving the board. Each time the beetle moves, the arrow in the square it leaves turns

$\frac{\pi}{2}$ clockwise. Prove that the beetle leaves the board in at most $2^{3n-1}(n-1)!-4$ years after it first moves.

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7 The convex quadrilateral $ABCD$ is inscribed in the circle S_1. Let O be the intersection of AC and BD. Circle S_2 passes through D and O, intersecting AD and CD at M and N, respectively. Lines OM and AB intersect at R, lines ON and BC intersect at T, and R and T lie on the same side of line BD as A.
Prove that O, R,T, and B are concyclic.

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8 There are n aborigines on an island. Any two of them are either friends or enemies. One day, the chieftain orders that all
citizens (including himself) make and wear a necklace with zero or more stones so that:
(i) given a pair of friends, there exists a color such that each has a stone of that color;
(ii) given a pair of enemies,there does not exist a color such that each a stone of that color.

(a) Prove that the aborigines can carry out the chieftains order.
(b) What is the minimum number of colors of stones required for the
aborigines to carry out the chieftains order?
