Belarus Team Selection Test 2009

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–	Test 1
1	Two equal circles S_1 and S_2 meet at two different points. The line ℓ intersects S_1 at points A, C and S_2 at points B, D respectively (the order on ℓ: A, B, C, D) . Define circles Γ_1 and Γ_2 as follows: both Γ_1 and Γ_2 touch S_1 internally and S_2 externally, both Γ_1 and Γ_2 line ℓ, Γ_1 and Γ_2 lie in the different halfplanes relatively to line ℓ. Suppose that Γ_1 and Γ_2 touch each other. Prove that $AB = CD$. I. Voronovich
2	Find all $n \in N$ for which the value of the expression $x^n + y^n + z^n$ is constant for all $x, y, z \in R$ such that $x + y + z = 0$ and $xyz = 1$. D. Bazylev
3	Points T, P, H lie on the side BC, AC, AB respectively of triangle ABC, so that BP and AT are angle bisectors and CH is an altitude of ABC. Given that the midpoint of CH belongs to the segment PT, find the value of $\cos A + \cos B$ I. Voronovich
4	Let x, y, z be integer numbers satisfying the equality $yx^2 + (y^2 - z^2)x + y(y - z)^2 = 0$ a) Prove that number xy is a perfect square. b) Prove that there are infinitely many triples (x, y, z) satisfying the equality. I.Voronovich
–	Test 2
1	Prove that any positive real numbers a,b,c satisfy the inequlaity $$\frac{1}{(a+b)b} + \frac{1}{(b+c)c} + \frac{1}{(c+a)a} \geq \frac{9}{2(ab+bc+ca)}$$ I.Voronovich
2	a) Prove that there is not an infinte sequence $(x_n), n = 1, 2, ...$ of positive real numbers satisfying the relation $x_{n+2} = \sqrt{x_{n+1}} - \sqrt{x_n}$, $\forall n \in N$ (*) b) Do there exist sequences satisfying (*) and containing arbitrary many terms? I.Voronovich

3 Given trapezoid $ABCD$ ($AD \parallel BC$) with $AD \perp AB$ and $T = AC \cap BD$. A circle centered at point O is inscribed in the trapezoid and touches the side CD at point Q. Let P be the intersection point (different from Q) of the side CD and the circle passing through T, Q and O. Prove that $TP \parallel AD$.

I. Voronovich

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4 Given a graph with n ($n \geq 4$) vertices . It is known that for any two vertices A and B there exists a vertex which is connected by edges both with A and B. Find the smallest possible numbers of edges in the graph.

E. Barabanov

– Test 3

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1 Find all functions $f : R \to R$ and $g : R \to R$ such that $f(x - f(y)) = xf(y) - yf(x) + g(x)$ for all real numbers x, y.

I.Voronovich

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2 Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

Proposed by John Cuya, Peru

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3 Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa$$

then $a = b = c$.

Proposed by Angelo Di Pasquale, Australia

– Test 4

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1 Prove that there exist many natural numbers n so that both roots of the quadratic equation $x^2 + (2 - 3n^2)x + (n^2 - 1)^2 = 0$ are perfect squares.

S. Kuzmich

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2 In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A and F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.

Proposed by Davood Vakili, Iran

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3 Let $n \in \mathbb{N}$ and A_n set of all permutations $(a_1, \dots, a_n)$ of the set $\{1, 2, \dots, n\}$ for which

$$k|2(a_1 + \cdots + a_k), \text{ for all } 1 \le k \le n.$$

Find the number of elements of the set A_n.

Proposed by Vidan Govedarica, Serbia

– Test 5

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1 Let M, N be the midpoints of the sides AD, BC respectively of the convex quadrilateral $ABCD$, $K = AN \cap BM$, $L = CM \cap DN$. Find the smallest possible $c \in R$ such that $S(MKNL) < c \cdot S(ABCD)$ for any convex quadrilateral $ABCD$.

I. Voronovich

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2 Let a_1, a_2, ..., a_n be distinct positive integers, $n \ge 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1$, $3a_2$, ..., $3a_n$.

Proposed by Mohsen Jamaali, Iran

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3 Find all real numbers a for which there exists a function $f : R \to R$ asuch that $x + f(y) = a(y + f(x))$ for all real numbers $x, y \in R$.

I.Voronovich

– Test 6

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1 In a triangle ABC, AM is a median, BK is a bisectrix, $L = AM \cap BK$. It is known that $BC = a, AB = c, a > c$.
Given that the circumcenter of triangle ABC lies on the line CL, find AC

I. Voronovich

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2 In the coordinate plane consider the set S of all points with integer coordinates. For a positive integer k, two distinct points A, $B \in S$ will be called k-*friends* if there is a point $C \in S$ such that the area of the triangle ABC is equal to k. A set $T \subset S$ will be called k-*clique* if every two points in T are k-friends. Find the least positive integer k for which there exits a k-clique with more than 200 elements.

Proposed by Jorge Tipe, Peru

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3 Let a_0, a_1, a_2, ... be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \ge 2^n$ for all $n \ge 0$.

Proposed by Morteza Saghafian, Iran

– Test 7

1 Denote by $\phi(n)$ for all $n \in \mathbb{N}$ the number of positive integer smaller than n and relatively prime to n. Also, denote by $\omega(n)$ for all $n \in \mathbb{N}$ the number of prime divisors of n. Given that $\phi(n)|n-1$ and $\omega(n) \leq 3$. Prove that n is a prime number.

2 Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, assume it does not lie on line AB. Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

Proposed by Charles Leytem, Luxembourg

3 Let $S = \{x_1, x_2, \dots, x_{k+l}\}$ be a $(k+l)$-element set of real numbers contained in the interval $[0, 1]$; k and l are positive integers. A k-element subset $A \subset S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{l} \sum_{x_j \in S \setminus A} x_j \right| \leq \frac{k+l}{2kl}$$

Prove that the number of nice subsets is at least $\dfrac{2}{k+l}\dbinom{k+l}{k}$.

Proposed by Andrey Badzyan, Russia

– Test 8

1 On R a binary algebraic operation "*" is defined which satisfies the following two conditions:
i) for all $a, b \in R$, there exists a unique $x \in R$ such that $x * a = b$ (write $x = b/a$)
ii) $(a * b) * c = (a * c) * (b * c)$ for all $a, b, c \in R$
a) Is this operation necesarily commutative (i.e. $a * b = b * a$ for all $a, b \in R$) ?
b) Prove that $(a/b)/c = (a/c)/(b/c)$ and $(a/b) * c = (a * c)/(b * c)$ for all $a, b, c \in R$.

A. Mirotin

2 Does there exist a convex pentagon $A_1A_2A_3A_4A_5$ and a point X inside it such that $XA_i = A_{i+2}A_{i+3}$ for all $i = 1, ..., 5$ (all indices are considered modulo 5) ?

I. Voronovich

3 a) Does there exist a function $f : N \to N$ such that $f(f(n)) = f(n+1) - f(n)$ for all $n \in N$?
b) Does there exist a function $f : N \to N$ such that $f(f(n)) = f(n+2) - f(n)$ for all $n \in N$?

I. Voronovich