Belarus Team Selection Test 2011

www.artofproblemsolving.com/community/c1201452

by parmenides51, orl, Amir Hossein

– Test 1

1 Find all real a such that there exists a function $f : R \to R$ satisfying the equation $f(\sin x) + af(\cos x) = \cos 2x$ for all real x.

I.Voronovich

2 Points L and H are marked on the sides AB of an acute-angled triangle ABC so that CL is a bisector and CH is an altitude. Let P, Q be the feet of the perpendiculars from L to AC and BC respectively. Prove that $AP \cdot BH = BQ \cdot AH$.

I. Gorodnin

3 Any natural number $n, n \geq 3$ can be presented in different ways as a sum several summands (not necessarily different). Find the greatest possible value of these summands.

Folklore

4 Given a $n \times n$ square table. Exactly one beetle sits in each cell of the table. At 12.00 all beetles creeps to some neighbouring cell (two cells are neighbouring if they have the common side). Find the greatest number of cells which can become empty (i.e. without beetles) if
a) $n = 8$
b) $n = 9$

Problem Committee of BMO 2011

– Test 2

1 Is it possible to arrange the numbers $1, 2, ..., 2011$ over the circle in some order so that among any 25 successive numbers at least 8 numbers are multiples of 5 or 7 (or both 5 and 7) ?

I. Gorodnin

2 Two different points X, Y are marked on the side AB of a triangle ABC so that $\frac{AX \cdot BX}{CX^2} = \frac{AY \cdot BY}{CY^2}$. Prove that $\angle ACX = \angle BCY$.

I.Zhuk

3 Find all functions $f : R \to R, g : R \to R$ satisfying the following equality $f(f(x + y)) = xf(y) + g(x)$ for all real x and y.

I. Gorodnin

4 Given nonzero real numbers a,b,c with $a+b+c=a^2+b^2+c^2=a^3+b^3+c^3$. $(*)$
a) Find $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\right)(a+b+c-2)$
b) Do there exist pairwise different nonzero a,b,c satisfying $(*)$?

D. Bazylev

– Test 3

1 Given natural number $a>1$ and different odd prime numbers $p_1,p_2,...,p_n$ with $a^{p_1}\equiv 1$ (mod p_2), $a^{p_2}\equiv 1$ (mod p_3), ..., $a^{p_n}\equiv 1$(mod p_1).
Prove that

a) $(a-1)\vdots p_i$ for some $i=1,..,n$
b) Can $(a-1)$ be divisible by $p_i$for exactly one i of $i=1,...,n$?

I. Bliznets

2 The external angle bisector of the angle A of an acute-angled triangle ABC meets the circumcircle of $\triangle\ ABC$ at point T. The perpendicular from the orthocenter H of $\triangle\ ABC$ to the line TA meets the line BC at point P. The line TP meets the circumcircce of $\triangle\ ABC$ at point D. Prove that $AB^2+DC^2=AC^2+BD^2$

A. Voidelevich

3 In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

Proposed by Gerhard Wginger, Austria

– Test 4

1 Let A be the sum of all 10 distinct products of the sides of a convex pentagon, S be the area of the pentagon.
a) Prove thas $S\le\frac{1}{5}A$.
b) Does there exist a constant $c<\frac{1}{5}$ such that $S\le cA$?

I.Voronovich

2 Find the least positive integer n for which there exists a set $\{s_1,s_2,\ldots,s_n\}$ consisting of n distinct positive integers such that

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\cdots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

Proposed by Daniel Brown, Canada

3 Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is n-good if $n | P(m) - P(k)$ implies $n | m - k$ for all integers m, k. We say that (a, b) is $very\ good$ if (a, b) is n-good for infinitely many positive integers n.
-**(a)** Find a pair (a, b) which is 51-good, but not very good.
-**(b)** Show that all 2010-good pairs are very good.

Proposed by Okan Tekman, Turkey

– Test 5

1 Let $g(n)$ be the number of all n-digit natural numbers each consisting only of digits $0, 1, 2, 3$ (but not nessesarily all of them) such that the sum of no two neighbouring digits equals 2. Determine whether $g(2010)$ and $g(2011)$ are divisible by 11.

I.Kozlov

2 Positive real a, b, c satisfy the condition

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} = 1 + \frac{1}{6}\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)$$

Prove that

$$\frac{a^3bc}{b+c} + \frac{b^3ca}{a+c} + \frac{c^3ab}{a+b} \geq \frac{1}{6}(ab+bc+ca)^2$$

I.Voronovich

3 Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Proposed by Christopher Bradley, United Kingdom

– Test 6

1 AB and CD are two parallel chords of a parabola. Circle S_1 passing through points A, B intersects circle S_2 passing through C, D at points E, F. Prove that if E belongs to the parabola, then F also belongs to the parabola.

I.Voronovich

2 Do they exist natural numbers m, x, y such that

$$m^2 + 25 \vdots (2011^x - 1007^y)?$$

S. Finskii

3 2500 chess kings have to be placed on a 100×100 chessboard so that

(i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);
(ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

Proposed by Sergei Berlov, Russia

– Test 7

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1 In an acute-angled triangle ABC, the orthocenter is H. I_H is the incenter of $\triangle\ BHC$. The bisector of $\angle BAC$ intersects the perpendicular from I_H to the side BC at point K. Let F be the foot of the perpendicular from K to AB. Prove that $2KF + BC = BH + HC$

A. Voidelevich

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2 Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m\left(2^{n+1} - 1\right).$$

Proposed by Angelo Di Pasquale, Australia

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3 Find all functions $f : R \to R$ such that for all real x, y with $y \neq 0$

$$f(x - f(x/y)) = xf(1 - f(1/y))$$

and
a) $f(1 - f(1)) \neq 0$
b) $f(1 - f(1)) = 0$

S. Kuzmich, I.Voronovich

– Test 8

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1 Find the least possible number of elements which can be deleted from the set $\{1, 2, ..., 20\}$ so that the sum of no two different remaining numbers is not a perfect square.

N. Sedrakian , I.Voronovich

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2 Let $A_1A_2 \dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \dots, P_n$ onto lines $A_1A_2, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for points $X_1, \dots, X_n$ on sides $A_1A_2, \dots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \dots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

if
a) $X_1, \dots, X_n$ are the midpoints of the corressponding sides,
b) $X_1, \dots, X_n$ are the feet of the corressponding altitudes,
c) $X_1, \dots, X_n$ are arbitrary points on the corressponding lines.

Modified version of IMO 2010 SL G3 (https://artofproblemsolving.com/community/c6h418634p236197
(it was question c)

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3 Let $x_1, \dots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \dots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

Proposed by Sergei Berlov, Ilya Bogdanov, Russia
