Belarus Team Selection Test 2015

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–	Test 1
1	Solve the equation in nonnegative integers a,b,c: $3^a+2^b+2015=3c!$ I.Gorodnin
2	All the numbers $1,2,...,9$ are written in the cells of a 3×3 table (exactly one number in a cell) . Per move it is allowed to choose an arbitrary 2×2 square of the table and either decrease by 1 or increase by 1 all four numbers of the square. After some number of such moves all numbers of the table become equal to some number a. Find all possible values of a. I.Voronovich
3	The incircle of the triangle ABC touches the sides AC and BC at points P and Q respectively. N and M are the midpoints of AC and BC respectively. Let $X = AM\cap BP, Y = BN\cap AQ$. Given C,X,Y are collinear, prove that CX is the angle bisector of the angle ACB. I. Gorodnin
4	Prove that $(a+b+c)^5\ge 81(a^2+b^2+c^2)abc$ for any positive real numbers a,b,c I.Gorodnin
–	Test 2
1	N numbers are marked in the set $\{1,2,...,2000\}$ so that any pair of the numbers $(1,2),(2,4),...,(1000,2000)$ contains at least one marked number. Find the least possible value of N. I.Gorodnin
2	In the sequence of digits $2,0,2,9,3,...$ any digit it equal to the last digit in the decimal representation of the sum of four previous digits. Do the four numbers $2,0,1,5$ in that order occur in the sequence? Folklore
3	Let the incircle of the triangle ABC touch the side AB at point Q. The incircles of the triangles QAC and QBC touch AQ, AC and BQ, BC at points P,T and D,F respectively. Prove that $PDFT$ is a cyclic quadrilateral.

I.Gorodnin

4 Find all pairs of polynomials $p(x), q(x) \in R[x]$ satisfying the equality $p(x^2) = p(x)q(1-x) + p(1-x)q(x)$ for all real x.

I.Voronovich

– Test 3

1 Do there exist numbers $a, b \in R$ and surjective function $f : R \to R$ such that $f(f(x)) = bxf(x) + a$ for all real x?

I.Voronovich

2 The medians AM and BN of a triangle ABC are the diameters of the circles ω_1 and ω_2. If ω_1 touches the altitude CH, prove that ω_2 also touches CH.

I. Gorodnin

3 Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n + 1$ smaller rectangles.

Proposed by Serbia

– Test 4

1 A circle intersects a parabola at four distinct points. Let M and N be the midpoints of the arcs of the circle which are outside the parabola. Prove that the line MN is perpendicular to the axis of the parabola.

I. Voronovich

2 Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

Proposed by Titu Andreescu, USA

3 Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.
Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove that no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

Proposed by Tamas Fleiner and Peter Pal Pach, Hungary

–	Test 5
1	Find all positive integers n such that $n = q(q^2 - q - 1) = r(2r + 1)$ for some primes q and r. B.Gilevich
2	Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC. *Proposed by Estonia*
3	Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has $$\|y^2 - P(x)\| \le 2\|x\| \quad \text{if and only if} \quad \|x^2 - P(y)\| \le 2\|y\|.$$ Determine all possible values of $P(0)$. *Proposed by Belgium*
–	Test 6
1	Let $n \ge 2$ be an integer, and let A_n be the set $$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\, 0 \le k < n\}.$$ Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n . *Proposed by Serbia*
2	Define the function $f : (0, 1) \to (0, 1)$ by $$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2} \\ x^2 & \text{if } x \ge \frac{1}{2} \end{cases}$$ Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that $$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$ *Proposed by Denmark*

3 Consider a fixed circle Γ with three fixed points $A, B,$ and C on it. Also, let us fix a real number $\lambda \in (0,1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

Proposed by Jack Edward Smith, UK

– Test 7

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1 We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m .

Proposed by Abbas Mehrabian, Iran

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2 Given a cyclic $ABCD$ with $AB = AD$. Points M and N are marked on the sides CD and BC, respectively, so that $DM + BN = MN$. Prove that the circumcenter of the triangle AMN belongs to the segment AC.

N.Sedrakian

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3 Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Proposed by Hong Kong

– Test 8

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1 Given $m, n \in N$ such that $M > n^{n-1}$ and the numbers $m+1, m+2, ..., m+n$ are composite. Prove that exist distinct primes $p_1, p_2, ..., p_n$ such that $M + k$ is divisible by p_k for any $k = 1, 2, ..., n$.

Tuymaada Olympiad 2004, C.A.Grimm. USA

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2 In a cyclic quadrilateral $ABCD$, the extensions of sides AB and CD meet at point P, and the extensions of sides AD and BC meet at point Q. Prove that the distance between the orthocenters of triangles APD and AQB is equal to the distance between the orthocenters of triangles CQD and BPC.

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3 Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m)+n\big) + f(m) = f(n) + f(3m) + 2014$$

for all integers m and n.

Proposed by Netherlands