Belarus Team Selection Test 2016

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–	Test 1
1	Prove for positive a, b, c that $$(a^2+\frac{b^2}{c^2})(b^2+\frac{c^2}{a^2})(c^2+\frac{a^2}{b^2}) \ge abc(a+\frac{1}{a})(b+\frac{1}{b})(c+\frac{1}{c})$$
2	Let A, B, C denote intersection points of diagonals A_1A_4 and A_2A_5, A_1A_6 and A_2A_7, A_1A_9 and A_2A_{10} of the regular decagon $A_1A_2...A_{10}$ respectively Find the angles of the triangle ABC
3	Solve the equation $p^3 - q^3 = pq^3 - 1$ in primes p, q.
4	There is a graph with 30 vertices. If any of 26 of its vertices with their outgoiing edges are deleted, then the remained graph is a connected graph with 4 vertices. What is the smallest number of the edges in the initial graph with 30 vertices?
–	Test 2
1	Let a, b, c, d, x, y denote the lengths of the sides AB, BC, CD, DA and the diagonals AC, BD of a cyclic quadrilateral $ABCD$ respectively. Prove that $$(\frac{1}{a}+\frac{1}{c})^2+(\frac{1}{b}+\frac{1}{d})^2 \ge 8(\frac{1}{x^2}+\frac{1}{y^2})$$
2	Find all real numbers a such that exists function $\mathbb{R} \to \mathbb{R}$ satisfying the following conditions: 1) $f(f(x)) = xf(x) - ax$ for all real x 2) f is not constant 3) f takes the value a
3	Point A, B are marked on the right branch of the hyperbola $y = \frac{1}{x}, x > 0$. The straight line l passing through the origin O is perpendicular to AB and meets AB and given branch of the hyperbola at points D and C respectively. The circle through A, B, C meets l at F. Find $OD : CF$

4 On a circle there are $2n+1$ points, dividing it into equal arcs ($n \geq 2$). Two players take turns to erase one point. If after one player's turn, it turned out that all the triangles formed by the remaining points on the circle were obtuse, then the player wins and the game ends.
Who has a winning strategy: the starting player or his opponent?

– Test 3

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1 Find all functions $f:\mathbb{R}\to\mathbb{R}, g:\mathbb{R}\to\mathbb{R}$ such that

$$f(x-2f(y)) = xf(y) - yf(x) + g(x)$$

for all real x, y

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2 Given a graph with $n \geq 4$ vertices. It is known that for any two of vertices there is a vertex connected with none of these two vertices.
Find the greatest possible number of the edges in the graph.

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3 Let D, E, F denote the tangent points of the incircle of ABC with sides BC, AC, AB respectively. Let M be the midpoint of the segment EF. Let L be the intersection point of the circle passing through D, M, F and the segment AB, K be the intersection point of the circle passing through D, M, E and the segment AC.
Prove that the circle passing through A, K, L touches the line BC

– Test 4

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1 Given real numbers a, b, c, d such that $\sin a + b > \sin c + d, a + \sin b > c + \sin d$, prove that $a + b > c + d$

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2 A point A_1 is marked inside an acute non-isosceles triangle ABC such that $\angle A_1AB = \angle A_1BC$ and $\angle A_1AC = \angle A_1CB$. Points B_1 and C_1 are defined same way. Let G be the gravity center if the triangle ABC.
Prove that the points A_1, B_1, C_1, G are concyclic.

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3 Solve the equation $2^a - 5^b = 3$ in positive integers a, b.

– Test 5

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1 Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

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2 Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

3 Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

– Test 6

1 a) Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$. (It is 2015 IMO Shortlist A2 (https://artofproblemsolving.com/community/c6h1268817p6621849))
b) The same question for if

$$f(x - f(y)) = f(f(x)) - f(y) - 2$$

for all integers x, y

2 Points B_1 and C_1 are marked respectively on the sides AB and AC of an acute isosceles triangle ABC($AB = AC$) such that $BB_1 = AC_1$. The points B, C and S lie in the same half-plane with respect to the line B_1C_1 so that $\angle SB_1C_1 = \angle SC_1B_1 = \angle BAC$
Prove that B, C, S are colinear if and only if the triangle ABC is equilateral.

3 Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

– Test 7

1 Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

2 Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

3 For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

– Test 8

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1 There are $n \geq 1$ cities on a horizontal line. Each city is guarded by a pair of stationary elephants, one just to the left and one just ot the right of the city, and facing away from it. The $2n$ elephants are of different sizes. If an elephant walks forward, it will knock aside any elephant that it approaches from behind, and in face-to-face meeting, the smaller elephant will be knocked aside. A moving elephant will keep walking in the same direction until it is knocked aside.
Show that there is a unique city with the property that if any of the other cities orders its elephants to walk, then that city will not be invaded by an elephant.

IMO 2015, Shortlist C1 (`https://artofproblemsolving.com/community/c6h1268873p6622370`), modified by G. Smith

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2 Let K and L be the centers of the excircles of a non-isosceles triangle ABC opposite B and C respectively. Let B_1 and C_1 be the midpoints of the sides AC and AB respectively Let M and N be symmetric to B and C about B_1 and C_1 respectively.
Prove that the lines KM and LN meet on BC.

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3 Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.
