2019 United States Ersatz Math Olympiad

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Day 1 May 23, 2020

1 Let $ABCD$ be a cyclic quadrilateral. A circle centered at O passes through B and D and meets lines BA and BC again at points E and F (distinct from A, B, C). Let H denote the orthocenter of triangle DEF. Prove that if lines AC, DO, EF are concurrent, then triangle ABC and EHF are similar.

Robin Son

2 Let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta : \mathbb{Z}[x] \to \mathbb{Z}[x]$ (i.e. functions taking polynomials to polynomials) such that

- for any polynomials $p, q \in \mathbb{Z}[x]$, $\theta(p+q) = \theta(p) + \theta(q)$;
- for any polynomial $p \in \mathbb{Z}[x]$, p has an integer root if and only if $\theta(p)$ does.

Carl Schildkraut

3 Consider an infinite grid $\mathcal{G}$ of unit square cells. A *chessboard polygon* is a simple polygon (i.e. not self-intersecting) whose sides lie along the gridlines of $\mathcal{G}$.

Nikolai chooses a chessboard polygon F and challenges you to paint some cells of $\mathcal{G}$ green, such that any chessboard polygon congruent to F has at least 1 green cell but at most 2020 green cells. Can Nikolai choose F to make your job impossible?

Nikolai Beluhov

Day 2 May 24, 2020

4 Prove that for any prime p, there exists a positive integer n such that

$$1^n + 2^{n-1} + 3^{n-2} + \cdots + n^1 \equiv 2020 \pmod{p}.$$

Robin Son

5 Let $\mathcal{P}$ be a regular polygon, and let $\mathcal{V}$ be its set of vertices. Each point in $\mathcal{V}$ is colored red, white, or blue. A subset of $\mathcal{V}$ is *patriotic* if it contains an equal number of points of each color, and a side of $\mathcal{P}$ is *dazzling* if its endpoints are of different colors.

Suppose that $\mathcal{V}$ is patriotic and the number of dazzling edges of $\mathcal{P}$ is even. Prove that there exists a line, not passing through any point in $\mathcal{V}$, dividing $\mathcal{V}$ into two nonempty patriotic subsets.

Ankan Bhattacharya

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6 Let ABC be an acute scalene triangle with circumcenter O and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Let X, Y, Z be the midpoints of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Lines AD and YZ intersect at P, lines BE and ZX intersect at Q, and lines CF and XY intersect at R.

Suppose that lines YZ and BC intersect at A', and lines QR and EF intersect at D'. Prove that the perpendiculars from A, B, C, O, to the lines QR, RP, PQ, $A'D'$, respectively, are concurrent.

Ankan Bhattacharya
