RMM 2019 Shortlist

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by parmenides51, Physicsknight, Vlados021, huricane

–	Algebra
A1	Determine all the functions $f:\mathbb{R}\mapsto\mathbb{R}$ satisfies the equation $f(a^2+ab+f(b^2))=af(b)+b^2+f(a^2)\,\forall a,b\in\mathbb{R}$
A2	Given a positive integer n, determine the maximal constant C_n satisfying the following condition: for any partition of the set $\{1,2,\ldots,2n\}$ into two n-element subsets A and B, there exist labellings $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ of A and B, respectively, such that $$(a_1-b_1)^2+(a_2-b_2)^2+\ldots+(a_n-b_n)^2\ge C_n.$$ *(B. Serankou, M. Karpuk)*
–	Combinatorics
C1	Let k and N be integers such that $k>1$ and $N>2k+1$. A number of N persons sit around the Round Table, equally spaced. Each person is either a knight (always telling the truth) or a liar (who always lies). Each person sees the nearest k persons clockwise, and the nearest k persons anticlockwise. Each person says: "I see equally many knights to my left and to my right." Establish, in terms of k and N, whether the persons around the Table are necessarily all knights. Sergey Berlov, Russia
C2	Fix an integer $n\ge 2$. A fairy chess piece *leopard* may move one cell up, or one cell to the right, or one cell diagonally down-left. A leopard is placed onto some cell of a $3n\times 3n$ chequer board. The leopard makes several moves, never visiting a cell twice, and comes back to the starting cell. Determine the largest possible number of moves the leopard could have made. Dmitry Khramtsov, Russia
C3	Fix an odd integer $n>1$. For a permutation p of the set $\{1,2,...,n\}$, let S be the number of pairs of indices (i,j), $1\le i\le j\le n$, for which $p_i+p_{i+1}+...+p_j$ is divisible by n. Determine the maximum possible value of S. Croatia
–	Geometry

G1 Let BM be a median in an acute-angled triangle ABC. A point K is chosen on the line through C tangent to the circumcircle of $\triangle BMC$ so that $\angle KBC = 90^\circ$. The segments AK and BM meet at J. Prove that the circumcenter of $\triangle BJK$ lies on the line AC.

Aleksandr Kuznetsov, Russia

G2 Let ABC be an acute-angled triangle. The line through C perpendicular to AC meets the external angle bisector of $\angle ABC$ at D. Let H be the foot of the perpendicular from D onto BC. The point K is chosen on AB so that $KH \parallel AC$. Let M be the midpoint of AK. Prove that $MC = MB + BH$.

Giorgi Arabidze, Georgia,

G3 Let ABC be an acute-angled triangle with $AB \neq AC$, and let I and O be its incenter and circumcenter, respectively. Let the incircle touch BC, CA and AB at D, E and F, respectively. Assume that the line through I parallel to EF, the line through D parallel toAO, and the altitude from A are concurrent. Prove that the concurrency point is the orthocenter of the triangle ABC.

Petar Nizic-Nikolac, Croatia

G4 ver.I Let Ω be the circumcircle of an acute-angled triangle ABC. Let D be the midpoint of the minor arc AB of Ω. A circle ω centered at D is tangent to AB at E. The tangents to ω through C meet the segment AB at K and L, where K lies on the segment AL. A circle Ω_1 is tangent to the segments AL, CL, and also to Ω at point M. Similarly, a circle Ω_2 is tangent to the segments BK, CK, and also to Ω at point N. The lines LM and KN meet at P. Prove that $\angle KCE = \angle LCP$.

Poland

G4 ver.II Let Ω be the circumcircle of an acute-angled triangle ABC. A point D is chosen on the internal bisector of $\angle ACB$ so that the points D and C are separated by AB. A circle ω centered at D is tangent to the segment AB at E. The tangents to ω through C meet the segment AB at K and L, where K lies on the segment AL. A circle Ω_1 is tangent to the segments AL, CL, and also to Ω at point M. Similarly, a circle Ω_2 is tangent to the segments BK, CK, and also to Ω at point N. The lines LM and KN meet at P. Prove that $\angle KCE = \angle LCP$.

Poland

G5 A quadrilateral $ABCD$ is circumscribed about a circle with center I. A point $P \neq I$ is chosen inside $ABCD$ so that the triangles PAB, PBC, PCD, and PDA have equal perimeters. A circle Γ centered at P meets the rays PA, PB, PC, and PD at A_1, B_1, C_1, and D_1, respectively. Prove that the lines PI, A_1C_1, and B_1D_1 are concurrent.

Ankan Bhattacharya, USA

– Number Theory

N1 Let p and q be relatively prime positive odd integers such that $1 < p < q$. Let A be a set of pairs of integers (a, b), where $0 \leq a \leq p-1, 0 \leq b \leq q-1$, containing exactly one pair from each of the sets

$$\{(a, b), (a+1, b+1)\}, \{(a, q-1), (a+1, 0)\}, \{(p-1, b), (0, b+1)\}$$

whenever $0 \leq a \leq p-2$ and $0 \leq b \leq q-2$. Show that A contains at least $(p-1)(q+1)/8$ pairs whose entries are both even.

Agnijo Banerjee and Joe Benton, United Kingdom

– Original Day 2 problems (removed due to leak)

original P4 Let there be an equilateral triangle ABC and a point P in its plane such that $AP < BP < CP$. Suppose that the lengths of segments AP, BP and CP uniquely determine the side of ABC. Prove that P lies on the circumcircle of triangle ABC.

original P5 Two ants are moving along the edges of a convex polyhedron. The route of every ant ends in its starting point, so that one ant does not pass through the same point twice along its way. On every face F of the polyhedron are written the number of edges of F belonging to the route of the first ant and the number of edges of F belonging to the route of the second ant. Is there a polyhedron and a pair of routes described as above, such that only one face contains a pair of distinct numbers?

Proposed by Nikolai Beluhov

original P6 Let $P(x)$ be a nonconstant complex coefficient polynomial and let $Q(x, y) = P(x) - P(y)$. Suppose that polynomial $Q(x, y)$ has exactly k linear factors unproportional two by tow (without counting repetitons). Let $R(x, y)$ be factor of $Q(x, y)$ of degree strictly smaller than k. Prove that $R(x, y)$ is a product of linear polynomials.

Note: The *degree* of nontrivial polynomial $\sum_m \sum_n c_{m,n} x^m y^n$ is the maximum of $m+n$ along all nonzero coefficients $c_{m,n}$. Two polynomials are *proportional* if one of them is the other times a complex constant.

Proposed by Navid Safaie