German National Olympiad 2020, Final Round

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– Day 1

1 Let k be a circle with center M and let B be another point in the interior of k. Determine those points V on k for which $\measuredangle BVM$ becomes maximal.

2 In ancient times there was a Celtic tribe consisting of several families. Many of these families were at odds with each other, so that their chiefs would not shake hands.
At some point at the annual meeting of the chiefs they found it even impossible to assemble four or more of them in a circle with each of them being willing to shake his neighbour's hand. To emphasize the gravity of the situation, the Druid collected three pieces of gold from each family. The Druid then let all those chiefs shake hands who were willing to. For each handshake of two chiefs he paid each of them a piece of gold as a reward.

Show that the number of pieces of gold collected by the Druid exceeds the number of pieces paid out by at least three.

3 Show that the equation

$$x(x+1)(x+2)\dots(x+2020)-1=0$$

has exactly one positive solution x_0, and prove that this solution x_0 satisfies

$$\frac{1}{2020!+10}<x_0<\frac{1}{2020!+6}.$$

– Day 2

4 Determine all positive integers n for which there exists a positive integer d with the property that n is divisible by d and n^2+d^2 is divisible by d^2n+1.

5 Let $a_1,a_2,\dots,a_{22}$ be positive integers with sum 59.
Prove the inequality

$$\frac{a_1}{a_1+1}+\frac{a_2}{a_2+1}+\dots+\frac{a_{22}}{a_{22}+1}<16.$$

6 The insphere and the exsphere opposite to the vertex D of a (not necessarily regular) tetrahedron $ABCD$ touch the face ABC in the points X and Y, respectively. Show that $\measuredangle XAB=\measuredangle CAY$.