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– Day 1

**P1** On the sides of  $\triangle ABC$  points  $P, Q \in AB$  ( $P$  is between  $A$  and  $Q$ ) and  $R \in BC$  are chosen. The points  $M$  and  $N$  are defined as the intersection point of  $AR$  with the segments  $CP$  and  $CQ$ , respectively. If  $BC = BQ$ ,  $CP = AP$ ,  $CR = CN$  and  $\angle BPC = \angle CRA$ , prove that  $MP + NQ = BR$ .

**P2** Let  $b_1, \dots, b_n$  be nonnegative integers with sum 2 and  $a_0, a_1, \dots, a_n$  be real numbers such that  $a_0 = a_n = 0$  and  $|a_i - a_{i-1}| \leq b_i$  for each  $i = 1, \dots, n$ . Prove that

$$\sum_{i=1}^n (a_i + a_{i-1})b_i \leq 2$$

I believe that the original problem was for nonnegative real numbers and it was a typo on the version of the exam paper we had but I'm not sure the inequality would hold

**P3** Let  $a_1 \in \mathbb{Z}$ ,  $a_2 = a_1^2 - a_1 - 1, \dots, a_{n+1} = a_n^2 - a_n - 1$ . Prove that  $a_{n+1}$  and  $2n + 1$  are coprime.

– Day 2

**P4** Are there positive integers  $m > 4$  and  $n$ , such that  
 a)  $\binom{m}{3} = n^2$ ;  
 b)  $\binom{m}{4} = n^2 + 9$ ?

**P5** There are  $n$  points in the plane, some of which are connected by segments. Some of the segments are colored in white, while the others are colored black in such a way that there exist a completely white as well as a completely black closed broken line of segments, each of them passing through every one of the  $n$  points exactly once. It is known that the segments  $AB$  and  $BC$  are white. Prove that it is possible to recolor the segments in red and blue in such a way that  $AB$  and  $BC$  are recolored as red, meaning that recoloring every white as red and every black as blue is not acceptable, and that there exist a completely red as well as a completely blue closed broken line of segments, each of them passing through every one of the  $n$  points exactly once.

**P6** Let  $f(x)$  be a nonconstant real polynomial. The sequence  $\{a_i\}_{i=1}^{\infty}$  of real numbers is strictly increasing and unbounded, as

$$a_{i+1} < a_i + 2020.$$

The integers  $\lfloor f(a_1) \rfloor, \lfloor f(a_2) \rfloor, \lfloor f(a_3) \rfloor, \dots$  are written consecutively in such a way that their digits form an infinite sequence of digits  $\{s_k\}_{k=1}^{\infty}$  (here  $s_k \in \{0, 1, \dots, 9\}$ ). If  $n \in \mathbb{N}$ , prove that among the numbers  $\overline{s_{n(k-1)+1}s_{n(k-1)+2} \cdots s_{nk}}$ , where  $k \in \mathbb{N}$ , all  $n$ -digit numbers appear.

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