Final Round - Korea 2020

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Day 1

P1 Let $ABCD$ be an isosceles trapezoid such that $AB \parallel CD$ and $\overline{AD}=\overline{BC}, \overline{AB}>\overline{CD}$. Let E be a point such that $\overline{EC}=\overline{AC}$ and $EC\perp BC$, and $\angle ACE<90^{\circ}$. Let Γ be a circle with center D and radius DA, and Ω be the circumcircle of triangle AEB. Suppose that Γ meets Ω again at $F(\neq A)$, and let G be a point on Γ such that $\overline{BF}=\overline{BG}$.
Prove that the lines EG, BD meet on Ω.

P2 There are 2020 groups, each of which consists of a boy and a girl, such that each student is contained in exactly one group. Suppose that the students shook hands so that the following conditions are satisfied:

- boys didn't shake hands with boys, and girls didn't shake hands with girls;
- in each group, the boy and girl had shake hands exactly once;
- any boy and girl, each in different groups, didn't shake hands more than once;
- for every four students in two different groups, there are at least three handshakes.

Prove that one can pick 4038 students and arrange them at a circular table so that every two adjacent students had shake hands.

P3 Find all $f:\mathbb{Q}_+\to\mathbb{R}$ such that
$$f(x)+f(y)+f(z)=1$$
holds for every positive rationals x,y,z satisfying $x+y+z+1=4xyz$.

Day 2

P4 Do there exist two positive reals α, β such that each positive integer appears exactly once in the following sequence?

$$2020, [\alpha], [\beta], 4040, [2\alpha], [2\beta], 6060, [3\alpha], [3\beta], \cdots$$

If so, determine all such pairs; if not, prove that it is impossible.

P5 Let ABC be an acute triangle such that $\overline{AB}=\overline{AC}$. Let M, L, N be the midpoints of segment BC, AM, AC, respectively. The circumcircle of triangle AMC, denoted by Ω, meets segment AB at $P(\neq A)$, and the segment BL at Q. Let O be the circumcenter of triangle BQC. Suppose that the lines AC and PQ meet at X, OB and LN meet at Y, and BQ and CO meets at Z. Prove that the points X, Y, Z are collinear.

P6 Find all positive integers n such that $6(n^4 - 1)$ is a square of an integer.