Spain Mathematical Olympiad 2020

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by Sumgato

–	Day 1
1	A polynomial $p(x)$ with real coefficients is said to be *almeriense* if it is of the form: $$p(x)=x^3+ax^2+bx+a$$ And its three roots are positive real numbers in arithmetic progression. Find all *almeriense* polynomials such that $p\left(\frac{7}{4}\right)=0$
2	Consider the succession of integers $\{f(n)\}_{n=1}^{\infty}$ defined as: • $f(1)=1$. • $f(n)=f(n/2)$ if n is even. • If $n>1$ odd and $f(n-1)$ odd, then $f(n)=f(n-1)-1$. • If $n>1$ odd and $f(n-1)$ even, then $f(n)=f(n-1)+1$. a) Compute $f(2^{2020}-1)$. b) Prove that $\{f(n)\}_{n=1}^{\infty}$ is not periodical, that is, there do not exist positive integers t and n_0 such that $f(n+t)=f(n)$ for all $n\geq n_0$.
3	To each point of $\mathbb{Z}^3$ we assign one of p colors. Prove that there exists a rectangular parallelepiped with all its vertices in $\mathbb{Z}^3$ and of the same color.
–	Day 2
4	Ana and Benito play a game which consists of 2020 turns. Initially, there are 2020 cards on the table, numbered from 1 to 2020, and Ana possesses an extra card with number 0. In the k-th turn, the player that doesn't possess card $k-1$ chooses whether to take the card with number k or to give it to the other player. The number in each card indicates its value in points. At the end of the game whoever has most points wins. Determine whether one player has a winning strategy or whether both players can force a tie, and describe the strategy.
5	In an acute-angled triangle ABC, let M be the midpoint of AB and P the foot of the altitude to BC. Prove that if $AC+BC=\sqrt{2}AB$, then the circumcircle of triangle BMP is tangent to AC.

6 Let S be a finite set of integers. We define $d_2(S)$ and $d_3(S)$ as:

• $d_2(S)$ is the number of elements $a \in S$ such that there exist $x, y \in \mathbb{Z}$ such that $x^2 - y^2 = a$ • $d_3(S)$ is the number of elements $a \in S$ such that there exist $x, y \in \mathbb{Z}$ such that $x^3 - y^3 = a$

(a) Let m be an integer and $S = \{m, m+1, \ldots, m+2019\}$. Prove:

$$d_2(S) > \frac{13}{7} d_3(S)$$

(b) Let $S_n = \{1, 2, \ldots, n\}$ with n a positive integer. Prove that there exists a N so that for all $n > N$:

$$d_2(S_n) > 4 \cdot d_3(S_n)$$
