Estonia Team Selection Test 2015

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–	Day 1	
1	Let n be a natural number, $n \geq 5$, and $a_1, a_2, ..., a_n$ real numbers such that all possible sums $a_i + a_j$, where $1 \leq i < j \leq n$, form $\frac{n(n-1)}{2}$ consecutive members of an arithmetic progression when taken in some order. Prove that $a_1 = a_2 = ... = a_n$.
2	A square-shaped pizza with side length 30 cm is cut into pieces (not necessarily rectangular). All cuts are parallel to the sides, and the total length of the cuts is 240 cm. Show that there is a piece whose area is at least 36 cm^2
3	Let q be a fixed positive rational number. Call number x *charismatic* if there exist a positive integer n and integers $a_1, a_2, ..., a_n$ such that $x = (q+1)^{a_1} \cdot (q+2)^{a_2} ... (q+n)^{a_n}$. a) Prove that q can be chosen in such a way that every positive rational number turns out to be charismatic. b) Is it true for every q that, for every charismatic number x, the number $x+1$ is charismatic, too?
–	Day 2
4	Altitudes AD and BE of an acute triangle ABC intersect at H. Let $C_1(H, HE)$ and $C_2(B, BE)$ be two circles tangent at AC at point E. Let $P \neq E$ be the second point of tangency of the circle $C_1(H, HE)$ with its tangent line going through point C, and $Q \neq E$ be the second point of tangency of the circle $C_2(B, BE)$ with its tangent line going through point C. Prove that points D, P, and Q are collinear.
5	Find all functions f from reals to reals which satisfy $f(f(x)+f(y)) = f(x^2)+2x^2f(y)+(f(y))^2$ for all real numbers x and y.
6	In any rectangular game board with black and white squares, call a row X a mix of rows Y and Z whenever each cell in row X has the same colour as either the cell of the same column in row Y or the cell of the same column in row Z. Let a natural number $m \geq 3$ be given. In some rectangular board, black and white squares lie in such a way that all the following conditions hold. 1) Among every three rows of the board, one is a mix of two others. 2) For every two rows of the board, their corresponding cells in at least one column have different colours. 3) For every two rows of the board, their corresponding cells in at least one column have equal

colours.
4) It is impossible to add a new row with each cell either black or white to the board in a way leaving both conditions 1) and 2) still in force
Find all possibilities of what can be the number of rows of the board.

– Day 3

7 Prove that for every prime number p and positive integer a, there exists a natural number n such that p^n contains a consecutive equal digits.

8 Find all positive integers n for which it is possible to partition a regular n-gon into triangles with diagonals not intersecting inside the n-gon such that at every vertex of the n-gon an odd number of triangles meet.

9 The orthocenter of an acute triangle ABC is H. Let K and P be the midpoints of lines BC and AH, respectively. The angle bisector drawn from the vertex A of the triangle ABC intersects with line KP at D. Prove that $HD \perp AD$.

– Day 4

10 Let n be an integer and a, b real numbers such that $n > 1$ and $a > b > 0$. Prove that

$$(a^n - b^n)\left(\frac{1}{b^{n-1}} - \frac{1}{a^{n-1}}\right) > 4n(n-1)(\sqrt{a} - \sqrt{b})^2$$

11 Let M be the midpoint of the side AB of a triangle ABC. A circle through point C that has a point of tangency to the line AB at point A and a circle through point C that has a point of tangency to the line AB at point B
intersect the second time at point N. Prove that $|CM|^2 + |CN|^2 - |MN|^2 = |CA|^2 + |CB|^2 - |AB|^2$.

12 Call an n-tuple $(a_1, ..., a_n)$ *occasionally periodic* if there exist a nonnegative integer i and a positive integer p satisfying $i + 2p \leq n$ and $a_{i+j} = a_{i+p+j}$ for every $j = 1, 2, ..., p$. Let k be a positive integer. Find the least positive integer n for which there exists an n-tuple $(a_1, ..., a_n)$ with elements from set $\{1, 2, ..., k\}$, which is not occasionally periodic but whose arbitrary extension $(a_1, ..., a_n, a_{n+1})$ is occasionally periodic for any $a_{n+1} \in \{1, 2, ..., k\}$.