Estonia Team Selection Test 2016

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by parmenides51

– Day 1

1 There are k heaps on the table, each containing a different positive number of stones. Juri and Mari make moves alternatingly, Juri starts. On each move, the player making the move has to pick a heap and remove one or more stones in it from the table; in addition, the player is allowed to distribute any number of remaining stones from that heap in any way between other non-empty heaps. The player to remove the last stone from the table wins. For which positive integers k does Juri have a winning strategy for any initial state that satisfies the conditions?

2 Let p be a prime number. Find all triples (a, b, c) of integers (not necessarily positive) such that $a^bb^cc^a = p$.

3 Find all functions $f : R \to R$ satisfying the equality $f(2^x + 2y) = 2^yf(f(x))f(y)$for every $x, y \in R$.

– Day 2

4 Prove that for any positive integer $n \geq, 2 \cdot \sqrt{3} \cdot \sqrt[3]{4} ... \sqrt[n-1]{n} > n$

5 Let O be the circumcentre of the acute triangle ABC. Let c_1 and c_2 be the circumcircles of triangles ABO and ACO. Let P and Q be points on c_1 and c_2 respectively, such that OP is a diameter of c_1 and OQ is a diameter of c_2. Let T be the intesection of the tangent to c_1 at P and the tangent to c_2 at Q. Let D be the second intersection of the line AC and the circle c_1. Prove that the points D, O and T are collinear

6 A circle is divided into arcs of equal size by n points ($n \geq 1$). For any positive integer x, let $P_n(x)$ denote the number of possibilities for colouring all those points, using colours from x given colours, so that any rotation of the colouring by $i \cdot \frac{360^o}{n}$, where i is a positive integer less than n, gives a colouring that differs from the original in at least one point. Prove that the function $P_n(x)$ is a polynomial with respect to x.

– Day 3

7 On the sides AB, BC and CA of triangle ABC, points L, M and N are chosen, respectively, such that the lines CL, AM and BN intersect at a common point O inside the triangle and the

quadrilaterals $ALON, BMOL$ and $CNOM$ have incircles. Prove that

$$\frac{1}{AL \cdot BM} + \frac{1}{BM \cdot CN} + \frac{1}{CN \cdot AL} = \frac{1}{AN \cdot BL} + \frac{1}{BL \cdot CM} + \frac{1}{CM \cdot AN}$$

8 Let x, y and z be positive real numbers such that $x+y+z = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. Prove that $xy+yz+zx \geq 3$.

9 Let n be a positive integer such that there exists a positive integer that is less than $\sqrt{n}$ and does not divide n. Let $(a_1, ..., a_n)$ be an arbitrary permutation of $1, ..., n$. Let $a_{i1} < ... < a_{ik}$ be its maximal increasing subsequence and let $a_{j1} > ... > a_{jl}$ be its maximal decreasing subsequence.
Prove that tuples $(a_{i1}, ..., a_{ik})$ and $(a_{j1}, ..., a_{jl})$ altogether contain at least one number that does not divide n.

– Day 4

10 Let m be an integer, $m \geq 2$. Each student in a school is practising m hobbies the most. Among any m students there exist two students who have a common hobby. Find the smallest number of students for which there must exist a hobby which is practised by at least 3 students .

11 Find all positive integers n such that $(n^2 + 11n - 4) \cdot n! + 33 \cdot 13^n + 4$ is a perfect square

12 The circles k_1 and k_2 intersect at points M and N. The line ℓ intersects with the circle k_1 at points A and C and with circle k_2 at points B and D, so that points A, B, C and D are on the line ℓ in that order. Let X be a point on line MN such that the point M is between points X and N. Lines AX and BM intersect at point P and lines DX and CM intersect at point Q. Prove that $PQ \parallel \ell$.