Estonia Team Selection Test 2017

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– Day 1

1 Do there exist two positive powers of 5 such that the number obtained by writing one after the other is also a power of 5?

2 Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left| \frac{a_i}{a_j} - \frac{a_k}{a_l} \right| \le C.$$

3 Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

– Day 2

4 Let ABC be an isosceles triangle with apex A and altitude AD. On AB, choose a point F distinct from B such that CF is tangent to the incircle of ABD. Suppose that $\triangle\ BCF$ is isosceles. Show that those conditions uniquely determine:
a) which vertex of BCF is its apex,
b) the size of $\angle BAC$

5 The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down $001, 111$ and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

6 Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy) \left(f(f(x^2)) + f(f(y^2)) \right).$$

– Day 3

7 Let n be a positive integer. In how many ways can an $n \times n$ table be filled with integers from 0 to 5 such that
a) the sum of each row is divisible by 2 and the sum of each column is divisible by 3
b) the sum of each row is divisible by 2, the sum of each column is divisible by 3 and the sum of each of the two diagonals is divisible by 6?

8 Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Proposed by Tigran Margaryan, Armenia

9 Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

– Day 4

10 Let ABC be a triangle with $AB = \frac{AC}{2} + BC$. Consider the two semicircles outside the triangle with diameters AB and BC. Let X be the orthogonal projection of A onto the common tangent line of those semicircles. Find $\angle CAX$.

11 For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Proposed by Warut Suksompong, Thailand

12 Let $n \geq 3$ be a positive integer. Find the maximum number of diagonals in a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.