Estonia Team Selection Test 2018

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–	Day 1
1	There are distinct points $O, A, B, K_1, ..., K_n, L_1, ..., L_n$ on a plane such that no three points are collinear. The open line segments $K_1L_1, ..., K_nL_n$ are coloured red, other points on the plane are left uncoloured. An allowed path from point O to point X is a polygonal chain with first and last vertices at points O and X, containing no red points. For example, for $n = 1$, and $K_1 = (-1, 0)$, $L_1 = (1, 0)$, $O = (0, -1)$, and $X = (0, 1)$, OK_1X and OL_1X are examples of allowed paths from O to X, there are no shorter allowed paths. Find the least positive integer n such that it is possible that the first vertex that is not O on any shortest possible allowed path from O to A is closer to B than to A, and the first vertex that is not O on any shortest possible allowed path from O to B is closer to A than to B.
2	Find the greatest number of depicted pieces composed of 4 unit squares that can be placed without overlapping on an $n \times n$ grid (where n is a positive integer) in such a way that it is possible to move from some corner to the opposite corner via uncovered squares (moving between squares requires a common edge). The shapes can be rotated and reflected. `https://cdn.artofproblemsolving.com/attachments/b/d/f2978a24fdd737edfafa5927a8d2129eb586e png`
3	Given a real number c and an integer $m, m \geq 2$. Real numbers $x_1, x_2, ..., x_m$ satisfy the conditions $x_1 + x_2 + ... + x_m = 0$ and $\frac{x_1^2+x_2^2+...+x_m^2}{m} = c$. Find max $(x_1, x_2, ..., x_m)$ if it is known to be as small as possible.
–	Day 2
4	Find all functions $f : R \to R$ that satisfy $f(xy + f(xy)) = 2xf(y)$ for all $x, y \in R$
5	Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.
6	We call a positive integer n whose all digits are distinct *bright*, if either n is a one-digit number or there exists a divisor of n which can be obtained by omitting one digit of n and which is bright itself. Find the largest bright positive integer. (We assume that numbers do not start with zero.)
–	Day 3

7 Let AD be the altitude ABC of an acute triangle. On the line AD are chosen different points E and F so that $|DE| = |DF|$ and point E is in the interior of triangle ABC. The circumcircle of triangle BEF intersects BC and BA for second time at points K and M respectively. The circumcircle of the triangle CEF intersects the CB and CA for the second time at points L and N respectively. Prove that the lines AD, KM and LN intersect at one point.

8 Find all integers $k \geq 5$ for which there is a positive integer n with exactly k positive divisors $1 = d_1 < d_2 < ... < d_k = n$ and $d_2d_3 + d_3d_5 + d_5d_2 = n$.

9 Let m and n be positive integers. Player A has a field of $m \times n$, and player B has a $1 \times n$ field (the first is the number of rows). On the first move, each player places on each square of his field white or black chip as he pleases. At each next on the move, each player can change the color of randomly chosen pieces on your field to the opposite, provided that in no row for this move will not change more than one chip (it is allowed not to change not a single chip). The moves are made in turn, player A starts. Player A wins if there is such a position that in the only row player B's squares, from left to right, are the same as in some row of player's field A. Prove that player A has the ability to win for any game of player B if and only if $n < 2m$.

– Day 4

10 A sequence of positive real numbers $a_1, a_2, a_3, ...$ satisfies $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. A sequence $b_1, b_2, b_3, ...$ is defined by equations $b_1 = a_1$, $b_n = a_n + (b_1 + b_3 + ... + b_{n-1})$ for even $n > 1$, $b_n = a_n + (b_2 + b_4 + ... + b_{n-1})$ for odd $n > 1$.
Prove that if $n \geq 3$, then $\frac{1}{3} < \frac{b_n}{n \cdot a_n} < 1$

11 Let k be a positive integer. Find all positive integers n, such that it is possible to mark n points on the sides of a triangle (different from its vertices) and connect some of them with a line in such a way that the following conditions are satisfied:
1) there is at least 1 marked point on each side,
2) for each pair of points X and Y marked on different sides, on the third side there exist exactly k marked points which are connected to both X and Y and exactly k points which are connected to neither X nor Y

12 We call the polynomial $P(x)$ simple if the coefficient of each of its members belongs to the set $\{-1, 0, 1\}$.
Let n be a positive integer, $n > 1$. Find the smallest possible number of terms with a non-zero coefficient in a simple n-th degree polynomial with all values at integer places are divisible by n.