Estonia Team Selection Test 2019

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– Day 1

1 Some positive integer n is written on the board. Andrey and Petya is playing the following game. Andrey finds all the different representations of the number n as a product of powers of prime numbers (values degrees greater than 1), in which each factor is greater than all previous or equal to the previous one. Petya finds all different representations of the number n as a product of integers greater than 1, in which each factor is divisible by all the previous factors. The one who finds more performances wins, if everyone finds the same number of representations, the game ends in a draw. Find all positive integers n for which the game will end in a draw.

Note.
The representation of the number n as a product is also considered a representation consisting of a single factor n.

2 In an acute-angled triangle ABC, the altitudes intersect at point H, and point K is the foot of the altitude drawn from the vertex A. Circle c passing through points A and K intersects sides AB and AC at points M and N, respectively. The line passing through point A and parallel to line BC intersects the circumcircles of triangles AHM and AHN for second time, respectively, at points X and Y. Prove that $|XY| = |BC|$.

3 Find all functions $f : R \to R$ which for all $x, y \in R$ satisfy $f(x^2)f(y^2) + |x|f(-xy^2) = 3|y|f(x^2y)$.

– Day 2

4 Let us call a real number r *interesting*, if $r = a + b\sqrt{2}$ for some integers a and b. Let $A(x)$ and $B(x)$ be polynomial functions with interesting coefficients for which the constant term of $B(x)$ is 1, and $Q(x)$ be a polynomial function with real coefficients such that $A(x) = B(x) \cdot Q(x)$. Prove that the coefficients of $Q(x)$ are interesting.

5 Boeotia is comprised of 3 islands which are home to 2019 towns in total. Each flight route connects three towns, each on a different island, providing connections between any two of them in both directions. Any two towns in the country are connected by at most one flight route. Find the maximal number of flight routes in the country

6 It is allowed to perform the following transformations in the plane with any integers a:
(1) Transform every point (x, y) to the corresponding point $(x + ay, y)$,
(2) Transform every point (x, y) to the corresponding point $(x, y + ax)$.

Does there exist a non-square rhombus whose all vertices have integer coordinates and which can be transformed to:
a) Vertices of a square,
b) Vertices of a rectangle with unequal side lengths?

– Day 3

7 An acute-angled triangle ABC has two altitudes BE and CF. The circle with diameter AC intersects the segment BE at point P. A circle with diameter AB intersects the segment CF at point Q and the extension of this altitude at point Q'. Prove that $\angle PQ'Q = \angle PQB$.

8 Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.
Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

9 Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the number of divisors of sn and of sk are equal.

– Day 4

10 Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, ..., n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

11 Given a circle ω with radius 1. Let T be a set of triangles good, if the following conditions apply:
(a) the circumcircle of each triangle in the set T is ω;
(b) The interior of any two triangles in the set T has no common point.
Find all positive real numbers t, for which for each positive integer n there is a good set of n triangles, where the perimeter of each triangle is greater than t.

12 Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers such that $a_0 = 0, a_1 = 1$, and for every $n \geq 2$ there exists $1 \leq k \leq n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximum possible value of $a_{2018} - a_{2017}$.