Saudi Arabia IMO Team Selection Test 2013

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by parmenides51, IstekOlympiadTeam

– Day I

1 Triangle ABC is inscribed in circle ω. Point P lies inside triangle ABC.Lines AP, BP and CP intersect ω again at points A_1, B_1 and C_1 (other than A, B, C), respectively. The tangent lines to ω at A_1 and B_1 intersect at C_2.The tangent lines to ω at B_1 and C_1 intersect at A_2. The tangent lines to ω at C_1 and A_1 intersect at B_2. Prove that the lines AA_2, BB_2 and CC_2 are concurrent.

2 Let $S = f\{0.1.2.3, ...\}$ be the set of the non-negative integers. Find all strictly increasing functions $f : S \to S$ such that $n + f(f(n)) \leq 2f(n)$ for every n in S

3 A Saudi company has two offices. One office is located in Riyadh and the other in Jeddah. To insure the connection between the two offices, the company has designated from each office a number of correspondents so that :
(a) each pair of correspondents from the same office share exactly one common correspondent from the other office.
(b) there are at least 10 correspondents from Riyadh.
(c) Zayd, one of the correspondents from Jeddah, is in contact with exactly 8 correspondents from Riyadh.
What is the minimum number of correspondents from Jeddah who are in contact with the correspondent Amr from Riyadh?

4 Determine whether it is possible to place the integers $1, 2, ..., 2012$ in a circle in such a way that the 2012 products of adjacent pairs of numbers leave pairwise distinct remainders when divided by 2013.

– Day II

1 Find the maximum and the minimum values of $S = (1 - x_1)(1 - y_1) + (1 - x_2)(1 - y_2)$ for real numbers x_1, x_2, y_1, y_2 with $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 2013$.

2 Let ABC be an acute triangle, and let AA_1, BB_1, and CC_1 be its altitudes. Segments AA_1 and B_1C_1 meet at point K. The perpendicular bisector of segment A_1K intersects sides AB and AC at L and M, respectively. Prove that points A, A_1, L, and M lie on a circle.

3 For a positive integer n, we consider all its divisors (including 1 and itself). Suppose that $p\%$ of these divisors have their unit digit equal to 3. (For example $n = 117$, has six divisors, namely

$1, 3, 9, 13, 39, 117$. Two of these divisors namely 3 and 13, have unit digits equal to 3. Hence for $n = 117$, $p = 33.33...$). Find, when n is any positive integer, the maximum possible value of p.

4 Determine if there exists an infinite sequence of positive integers $a_1, a_2, a_3, ...$ such that
(i) each positive integer occurs exactly once in the sequence, and
(ii) each positive integer occurs exactly once in the sequence $|a_1 - a_2|, |a_2 - a_3|, ..., |a + k - a_{k+1}|, ...$

– Day III

1 Adel draws an $m \times n$ grid of dots on the coordinate plane, at the points of integer coordinates (a, b) where $1 \le a \le m$ and $1 \le b \le n$. He proceeds to draw a closed path along k of these dots, (a_1, b_1),(a_2, b_2),...,(a_k, b_k), such that (a_i, b_i) and (a_{i+1}, b_{i+1}) (where $(a_{k+1}, b_{k+1}) = (a_1, b_1)$) are 1 unit apart for each $1 \le i \le k$. Adel makes sure his path does not cross itself, that is, the k dots are distinct. Find, with proof, the maximum possible value of k in terms of m and n.

2 Given an integer $n \ge 2$, determine the number of ordered n-tuples of integers $(a_1, a_2, ..., a_n)$ such that
(a) $a_1 + a_2 + .. + a_n \ge n^2$ and
(b) $a_1^2 + a_2^2 + ... + a_n^2 \le n^3 + 1$

3 Let ABC be an acute triangle, M be the midpoint of BC and P be a point on line segment AM. Lines BP and CP meet the circumcircle of ABC again at X and Y , respectively, and sides AC at D and AB at E, respectively. Prove that the circumcircles of AXD and AYE have a common point $T \ne A$ on line AM.

4 Find all polynomials $p(x)$ with integer coefficients such that for each positive integer n, the number $2^n - 1$ is divisible by $p(n)$.