Saudi Arabia Team Selection Test for Balkan Math Olympiad 2013

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– Day I

1 The set G is defined by the points (x, y) with integer coordinates, $1 \le x \le 5$ and $1 \le y \le 5$. Determine the number of five-point sequences $(P_1, P_2, P_3, P_4, P_5)$ such that for $1 \le i \le 5$, $P_i = (x_i, i)$ is in G and $|x_1 - x_2| = |x_2 - x_3| = |x_3 - x_4| = |x_4 - x_5| = 1$.

2 For positive integers a and b, $gcd(a, b)$ denote their greatest common divisor and $lcm(a, b)$ their least common multiple. Determine the number of ordered pairs (a,b) of positive integers satisfying the equation $ab + 63 = 20\, lcm(a, b) + 12\, gcd(a, b)$

3 Solve the following equation where x is a real number: $\lfloor x^2 \rfloor - 10\lfloor x \rfloor + 24 = 0$

4 $ABCDEF$ is an equiangular hexagon of perimeter 21. Given that $AB = 3, CD = 4$, and $EF = 5$, compute the area of hexagon $ABCDEF$.

5 Let k be a real number such that the product of real roots of the equation

$$X^4 + 2X^3 + (2 + 2k)X^2 + (1 + 2k)X + 2k = 0$$

is -2013. Find the sum of the squares of these real roots.

6 Let ABC be a triangle with incenter I, and let D, E, F be the midpoints of sides BC, CA, AB, respectively. Lines BI and DE meet at P and lines CI and DF meet at Q. Line PQ meets sides AB and AC at T and S, respectively. Prove that $AS = AT$

7 Ayman wants to color the cells of a 50×50 chessboard into black and white so that each 2×3 or 3×2 rectangle contains an even number of white cells. Determine the number of ways Ayman can color the chessboard.

8 Prove that the ratio

$$\frac{1^1 + 3^3 + 5^5 + ... + (2^{2013} - 1)^{(2^{2013} - 1)}}{2^{2013}}$$

is an odd integer.

– Day II

1 In triangle ABC, $AB = AC = 3$ and $\angle A = 90^o$. Let M be the midpoint of side BC. Points D and E lie on sides AC and AB respectively such that $AD > AE$ and $ADME$ is a cyclic quadrilateral. Given that triangle EMD has area 2, find the length of segment CD.

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2 Find all functions $f : R \to R$ which satisfy for all $x, y \in R$ the relation $f(f(f(x) + y) + y) = x + y + f(y)$

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3 Find all positive integers x, y, z such that $2^x + 21^y = z^2$

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4 Ten students are standing in a line. A teacher wants to place a hat on each student. He has two colors of hats, red and white, and he has 10 hats of each color. Determine the number of ways in which the teacher can place hats such that among any set of consecutive students, the number of students with red hats and the number of students with blue hats differ by at most 2

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5 We call a positive integer [i]good[/i] if it doesnt have a zero digit and the sum of the squares of its digits is a perfect square. For example, 122 and 34 are good and 304 and 12 are not not good. Prove that there exists a n-digit good number for every positive integer n.

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6 Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$a\sqrt{b^2 + c^2 + bc} + b\sqrt{c^2 + a^2 + ca} + c\sqrt{a^2 + b^2 + ab} \geq \sqrt{3}$$

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7 The excircle ω_B of triangle ABC opposite B touches side AC, rays BA and BC at B_1, C_1 and A_1, respectively. Point D lies on major arc A_1C_1 of ω_B. Rays DA_1 and C_1B_1 meet at E. Lines AB_1 and BE meet at F. Prove that line FD is tangent to ω_B (at D).

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8 A social club has 101 members, each of whom is fluent in the same 50 languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let A be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of A.

– Day III

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1 $ABCD$ is a cyclic quadrilateral and ω its circumcircle. The perpendicular line to AC at D intersects AC at E and ω at F. Denote by ℓ the perpendicular line to BC at F. The perpendicular line to ℓ at A intersects ℓ at G and ω at H. Line GE intersects FH at I and CD at J. Prove that points C, F, I and J are concyclic

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2 Define Fibonacci sequence $\{F\}_{n=0}^{\infty}$ as $F_0 = 0, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for every integer $n > 1$. Determine all quadruples (a, b, c, n) of positive integers with a $< b < c$ such that each of $a, b, c, a + n, b + n, c + 2n$ is a term of the Fibonacci sequence.

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3 Let T be a real number satisfying the property:
For any nonnegative real numbers a, b, c, d, e with their sum equal to 1, it is possible to arrange them around a circle such that the products of any two neighboring numbers are no greater than T.
Determine the minimum value of T.

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4 Let $f : Z_{\geq 0} \to Z_{\geq 0}$ be a function which satisfies for all integer $n \geq 0$:
(a) $f(2n+1)^2 - f(2n)^2 = 6f(n) + 1$, (b) $f(2n) \geq f(n)$
where $Z_{\geq 0}$ is the set of nonnegative integers. Solve the equation $f(n) = 1000$

– Day IV

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1 $ABCD$ is a cyclic quadrilateral such that $AB = BC = CA$. Diagonals AC and BD intersect at E. Given that $BE = 19$ and $ED = 6$, find the possible values of AD.

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2 The base-7 representation of number n is $\overline{abc}_{(7)}$, and the base-9 representation of number n is $\overline{cba}_{(9)}$. What is the decimal (base-10) representation of n?

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3 Find the area of the set of points of the plane whose coordinates (x, y) satisfy $x^2 + y^2 \leq 4|x| + 4|y|$.

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4 Find all positive integers $n < 589$ for which 589 divides $n^2 + n + 1$.
