Saudi Arabia Team Selection Test for Balkan Math Olympiad 2015

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– Day I

1 Prove that for any integer $n \ge 2$, there exists a unique finite sequence $x_0, x_1, ..., x_n$ of real numbers which satisfies $x_0 = x_n = 0$ and $x_{i+1} - 8x_i^3 - 4x_i + 3x_{i-1} + 1 = 0$ for all $i = 1, 2, ..., n-1$. Prove moreover that $|x_i| \le \frac{1}{2}$ for all $i = 1, 2, ..., n-1$.

Nguy n Duy Thi Sn

2 Given 2015 subsets $A_1, A_2, ..., A_{2015}$ of the set $\{1, 2, ..., 1000\}$ such that $|A_i| \ge 2$ for every $i \ge 1$ and $|A_i \cap A_j| \ge 1$ for every $1 \le i < j \le 2015$. Prove that $k = 3$ is the smallest number of colors such that we can always color the elements of the set $\{1, 2, ..., 1000\}$ by k colors with the property that the subset A_i has at least two elements of different colors for every $i \ge 1$.

L Anh Vinh

3 Let ABC be a triangle, Γ its circumcircle, I its incenter, and ω a tangent circle to the line AI at I and to the side BC. Prove that the circles Γ and ω are tangent.

Malik Talbi

4 Let $n \ge 2$ be an integer and $p_1 < p_2 < ... < p_n$ prime numbers. Prove that there exists an integer k relatively prime with $p_1p_2...p_n$ and such that $gcd(k + p_1p_2...p_i, p_1p_2...p_n) = 1$ for all $i = 1, 2, ..., n-1$.

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– Day II

1 Find all strictly increasing functions $f : Z \to R$ such that for any $m, n \in Z$ there exists a $k \in Z$ such that $f(k) = f(m) - f(n)$.

Nguy n Duy Thi Sn

2 Find the number of 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ of distinct positive integers satisfying the following two conditions:
(a) $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 30$
(b) We can write $a_1, a_2, a_3, a_4, a_5, a_6$ on sides of a hexagon such that after a finite number of time choosing a vertex of the hexagon and adding 1 to the two numbers written on two sides adjacent to the vertex, we obtain
a hexagon with equal numbers on its sides.

L Anh Vinh

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3 Let ABC be a triangle, H_a, H_b and H_c the feet of its altitudes from A, B and C, respectively, T_a, T_b, T_c its touchpoints of the incircle with the sides BC, CA and AB, respectively. The circumcircles of triangles AH_bH_c and AT_bT_c intersect again at A'. The circumcircles of triangles BH_cH_a and BT_cT_a intersect again at B'. The circumcircles of triangles CH_aH_b and CT_aT_b intersect again at C'. Prove that the points A', B', C' are collinear.

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4 Prove that there exist infinitely many non prime positive integers n such that $7^{n-1} - 3^{n-1}$ is divisible by n.

L Anh Vinh
