**Saudi Arabia IMO Team Selection Test 2015**

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by parmenides51

– Day I

1 Find all functions $f : R_{>0} \to R$ such that $f\left(\frac{x}{y}\right) = f(x) + f(y) - f(x)f(y)$ for all $x, y \in R_{>0}$. Here, $R_{>0}$ denotes the set of all positive real numbers.

Nguy n Duy Thi Sn

2 Let $ABC$ be a triangle with orthocenter $H$. Let $P$ be any point of the plane of the triangle. Let $\Omega$ be the circle with the diameter $AP$ . The circle $\Omega$ cuts $CA$ and $AB$ again at $E$ and $F$ , respectively. The line $PH$ cuts $\Omega$ again at $G$. The tangent lines to $\Omega$ at $E, F$ intersect at $T$. Let $M$ be the midpoint of $BC$ and $L$ be the point on $MG$ such that $AL$ and $MT$ are parallel. Prove that $LA$ and $LH$ are orthogonal.

L Phc L

3 Let $n$ and $k$ be two positive integers. Prove that if $n$ is relatively prime with $30$, then there exist two integers $a$ and $b$, each relatively prime with $n$, such that $\frac{a^2-b^2+k}{n}$ is an integer.

Malik Talbi

– Day II

1 Let $ABC$ be an acute-angled triangle inscribed in the circle $(O)$, $H$ the foot of the altitude of $ABC$ at $A$ and $P$ a point inside $ABC$ lying on the bisector of $\angle BAC$. The circle of diameter $AP$ cuts $(O)$ again at $G$. Let $L$ be the projection of $P$ on $AH$. Prove that if $GL$ bisects $HP$ then $P$ is the incenter of the triangle $ABC$.

L Phc L

2 Hamza and Majid play a game on a horizontal $3 \times 2015$ white board. They alternate turns, with Hamza going first. A legal move for Hamza consists of painting three unit squares forming a horizontal $1 \times 3$ rectangle. A legal move for Majid consists of painting three unit squares forming a vertical $3 \times 1$ rectangle. No one of the two players is allowed to repaint already painted squares. The last player to make a legal move wins. Which of the two players, Hamza or Majid, can guarantee a win no matter what strategy his opponent chooses and what is his strategy to guarantee a win?

L Anh Vinh

**3** Let $a_1, a_2, ..., a_n$ be positive real numbers such that

$$a_1 + a_2 + ... + a_n = a_1^2 + a_2^2 + ... + a_n^2$$

Prove that

$$\sum_{1 \le i < j \le n} a_i a_j (1 - a_i a_j) \ge 0$$

V Quc B Cn.

**–** Day III

**1** Let $S$ be a positive integer divisible by all the integers $1, 2, ..., 2015$ and $a_1, a_2, ..., a_k$ numbers in $\{1, 2, ..., 2015\}$ such that $2S \le a_1 + a_2 + ... + a_k$. Prove that we can select from $a_1, a_2, ..., a_k$ some numbers so that the sum of these selected numbers is equal to $S$.

L Anh Vinh

**2** Let $ABC$ be a triangle and $\omega$ its circumcircle. Point $D$ lies on the arc $BC$ (not containing $A$) of $\omega$ and is different from $B, C$ and the midpoint of arc $BC$ . The tangent line to $\omega$ at $D$ intersects lines $BC, CA, AB$ at $A', B', C'$ respectively. Lines $BB'$ and $CC'$ intersect at $E$. Line $AA'$ intersects again circle $\omega$ at $F$. Prove that the three points $D, E, F$ are colinear.

Malik Talbi

**3** Find the number of binary sequences $S$ of length $2015$ such that for any two segments $I_1, I_2$ of $S$ of the same length, we have
The sum of digits of $I_1$ differs from the sum of digits of $I_2$ by at most $1$,
If $I_1$ begins on the left end of S then the sum of digits of $I_1$ is not greater than the sum of digits of $I_2$,
If $I_2$ ends on the right end of S then the sum of digits of $I_2$ is not less than the sum of digits of $I_1$.

L Anh Vinh

**–** Day IV

**1** Let $a, b, c, d$ be positive integers such that $ac + bd$ is divisible by $a^2 + b^2$. Prove that $gcd(c^2 + d^2, a^2 + b^2) > 1$.

Trn Nam Dng

**2** The total number of languages used in KAUST is $n$. For each positive integer $k \le n$, let $A_k$ be the set of all those people in KAUST who can speak at least $k$ languages; and let $B_k$ be the set of all people $P$ in KAUST with the property that, for any $k$ pairwise different languages (used in KAUST), $P$ can speak at least one of these $k$ languages. Prove that

(a) If $2k \geq n+1$ then $A_k \subseteq B_k$
(b) If $2k \leq n+1$ then $A_k \supseteq B_k$.

Nguy n Duy Thi Sn

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**3** Let $a, b, c$ be positive real numbers satisfying the condition

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=10$$

Find the greatest value and the least value of

$$T=(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$$

Trn Nam Dng

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