Saudi Arabia Team Selection Test for Balkan Math Olympiad 2017

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– Day I

1 Let $n = p_1p_2...p_{2017}$ be the positive integer where $p_1, p_2, ..., p_{2017}$ are 2017 distinct odd primes. A triangle is called *nice* if it is a right triangle with integer side lengths and the inradius is n. Find the number of nice triangles (two triangles are consider different if their tuples of length of sides are different)

2 Let ABC be an acute triangle with AT, AS respectively are the internal, external angle bisector of ABC and $T, S \in BC$. On the circle with diameter TS, take an arbitrary point P that lies inside the triangle ABC. Denote D, E, F, I as the incenter of triangle PBC, PCA, PAB, ABC. Prove that four lines AD, BE, CF and IP are concurrent.

3 How many ways are there to insert plus signs $+$ between the digits of number $111111...111$ which includes thirty of digits 1 so that the result will be a multiple of 30?

4 Fibonacci sequences is defined as $f_1 = 1$,$f_2 = 2$, $f_{n+1} = f_n + f_{n-1}$ for $n \ge 2$.
a) Prove that every positive integer can be represented as sum of several distinct Fibonacci number.
b) A positive integer is called *Fib-unique* if the way to represent it as sum of several distinct Fibonacci number is unique. Example: 13 is not Fib-unique because $13 = 13 = 8+5 = 8+3+2$. Find all Fib-unique.

– Day II

1 Find the smallest prime q such that

$$q = a_1^2 + b_1^2 = a_2^2 + 2b_2^2 = a_3^2 + 3b_3^2 = ... = a_{10}^2 + 10b_{10}^2$$

where $a_i, b_i (i = 1, 2, ..., 10)$ are positive integers

2 Polynomial P(x) with integer coefficient is called *cube-presented* if it can be represented as sum of several cube of polynomials with integer coefficients.
Examples: $3x + 3x^2$ is cube-represented because $3x + 3x^2 = (x + 1)^3 + (-x)^3 + (-1)^3$.
a) Is $3x^2$ a cube-represented polynomial?
b). How many quadratic polynomial P(x) with integer coefficients belong to the set $\{1, 2, 3, ..., 2017\}$ which is cube-represented?

3 We put four numbers $1, 2, 3, 4$ around a circle in order. One starts at the number 1 and every step, he moves to an adjacent number on either side. How many ways he can move such that sum of the numbers he visits in his path (including the starting number) is equal to 21?

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4 Let ABC be a triangle with A is an obtuse angle. Denote BE as the internal angle bisector of triangle ABC with $E \in AC$ and suppose that $\angle AEB = 45^o$. The altitude AD of triangle ABC intersects BE at F. Let O_1, O_2 be the circumcenter of triangles FED, EDC. Suppose that EO_1, EO_2 meet BC at G, H respectively. Prove that $\frac{GH}{GB} = \tan \frac{a}{2}$

– Mock BMO

– Day I

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1 Prove that there are infinitely many positive integer n such that $n!$ is divisible by $n^3 - 1$.

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2 Let R^+ be the set of positive real numbers. Find all function $f : R^+ \to R$ such that, for all positive real number x and y, the following conditions are satisfied:
i) $2f(x) + 2f(y) \le f(x + y)$
ii) $(x + y)[yf(x) + xf(y)] \ge xyf(x + y)$

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3 Let ABC be an acute triangle and (O) be its circumcircle. Denote by H its orthocenter and I the midpoint of BC. The lines BH, CH intersect AC, AB at E, F respectively. The circles (IBF) and (ICE) meet again at D.
a) Prove that D, I, A are collinear and HD, EF, BC are concurrent.
b) Let L be the foot of the angle bisector of $\angle BAC$ on the side BC. The circle (ADL) intersects (O) again at K and intersects the line BC at S out of the side BC. Suppose that AK, AS intersects the circles (AEF) again at G, T respectively. Prove that $TG = TD$.

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4 Consider the set $X = \{1, 2, 3, ..., 2018\}$.
How many positive integers k with $2 \le k \le 2017$ that satisfy the following conditions:
i) There exists some partition of the set X into 1009 disjoint pairs which are $(a_1, b_1), (a_2, b_2), ..., (a_{1009}, b_{1009})$ with $|a_i - b_i| \in \{1, k\}$.
ii) For all partitions satisfy the condition (i), the sum $T = \sum_{i=1}^{1009} |a_i - b_i|$ has the right most digit is 9

– Day II

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1 Let a, b, c be positive real numbers. Prove that

$$\frac{a(b^2 + c^2)}{(b + c)(a^2 + bc)} + \frac{b(c^2 + a^2)}{(c + a)(b^2 + ca)} + \frac{c(a^2 + b^2)}{(a + b)(c^2 + ab)} \ge \frac{3}{2}$$

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2 Solve the following equation in positive integers x, y: $x^{2017} - 1 = (x-1)(y^{2015} - 1)$

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3 Let $ABCD$ be a cyclic quadrilateral and triangles ACD, BCD are acute. Suppose that the lines AB and CD meet at S. Denote by E the intersection of AC, BD. The circles (ADE) and (BCE) meet again at F.
a) Prove that $SF \perp EF.$
b) The point G is taken out side of the quadrilateral $ABCD$ such that triangle GAB and FDC are similar. Prove that $GA + FB = GB + FA$

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4 Let p be a prime number and a table of size $(p^2+p+1) \times (p^2+p+1)$ which is divided into unit cells. The way to color some cells of this table is called nice if there are no four colored cells that form a rectangle (the sides of rectangle are parallel to the sides of given table).
1. Let k be the number of colored cells in some nice coloring way. Prove that $k \le (p+1)(p^2+p+1)$. Denote this number as k_{max}.
2. Prove that all ordered tuples (a, b, c) with $0 \le a, b, c < p$ and $a+b+c > 0$ can be partitioned into p^2+p+1 sets $S_1, S_2, ...S_{p^2+p+1}$ such that two tuples (a_1, b_1, c_1) and (a_2, b_2, c_2) belong to the same set if and only if $a_1 \equiv ka_2, b_1 \equiv kb_2, c_1 \equiv kc_2$ (mod p) for some $k \in \{1, 2, 3, ..., p-1\}$.
3. For $1 \le i, j \le p^2+p+1$, if there exist $(a_1, b_1, c_1) \in S_i$ and $(a_2, b_2, c_2) \in S_j$ such that $a_1a_2 + b_1b_2 + c_1c_2 \equiv 0$ (mod p), we color the cell (i, j) of the given table. Prove that this coloring way is nice with k_{max} colored cells

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