Saudi Arabia Team Selection Test for Balkan Math Olympiad 2018

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– Day I

1 Find the smallest positive integer n which can not be expressed as $n = \frac{2^a-2^b}{2^c-2^d}$ for some positive integers a, b, c, d

2 Find all functions $f : R \to R$ such that $f(2x^3 + f(y)) = y + 2x^2 f(x)$ for all real numbers x, y.

3 The partition of $2n$ positive integers into n pairs is called *square-free* if the product of numbers in each pair is not a perfect square.Prove that if for $2n$ distinct positive integers, there exists one square-free partition, then there exists at least $n!$ square-free partitions.

4 Let ABC be an acute, non isosceles with I is its incenter. Denote D, E as tangent points of (I) on AB, AC, respectively. The median segments respect to vertex A of triangles ABE and ACD meet(I) atP, Q, respectively. Take points M, N on the line DE such that $AM \perp BE$ and $AN \perp CD$ respectively.
a) Prove that A lies on the radical axis of (MIP) and (NIQ).
b) Suppose that the orthocenter H of triangle ABC lies on (I). Prove that there exists a line which is tangent to three circles of center A, B, C and all pass through H.

– Day II

1 Let ABC be a triangle with M, N, P as midpoints of the segments BC, CA, AB respectively. Suppose that I is the intersection of angle bisectors of $\angle BPM, \angle MNP$ and J is the intersection of angle bisectors of $\angle CNM, \angle MPN$. Denote (ω_1) as the circle of center I and tangent to MP at D, (ω_2) as the circle of center J and tangent to MN at E.
a) Prove that DE is parallel to BC.
b) Prove that the radical axis of two circles $(\omega_1), (\omega_2)$ bisects the segment DE.

2 Suppose that 2018 numbers 1 and -1 are written around a circle. For every two adjacent numbers, their product is taken. Suppose that the sum of all 2018 products is negative. Find all possible values of sum of 2018 given numbers.

3 Find all positive integers n such that $\phi(n)$ is a divisor of $n^2 + 3$.

4 Find all functions $f : Z \to Z$ such that $xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{(f(x))^2}{x} + f(yf(y))$, for all $x, y \in Z$, $x \neq 0$.