Saudi Arabia Team Selection Test for Balkan Math Olympiad 2019

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– Day I

1 Let p be an odd prime number.
a) Show that p divides $n2^n + 1$ for infinitely many positive integers n.
b) Find all n satisfy condition above when $p = 3$

2 Let Ibe the incenter of triangle $ABC$and J the excenter of the side BC: Let M be the midpoint of CB and N the midpoint of arc BAC of circle (ABC). If T is the symmetric of the point N by the point A, prove that the quadrilateral $JMIT$ is cyclic

3 For $n \ge 3$, it is given an $2n \times 2n$ board with black and white squares. It is known that all border squares are black and no 2×2 subboard has all four squares of the same color. Prove that there exists a 2×2 subboard painted like a chessboard, i.e. with two opposite black corners and two opposite white corners.

– Day II

1 There are n people with hats present at a party. Each two of them greeted each other exactly once and each greeting consisted of exchanging the hats that the two persons had at the moment. Find all $n \ge 2$ for which the order of
greetings can be arranged in such a way that after all of them, each person has their own hat back.

2 Let sequences of real numbers (x_n) and (y_n) satisfy $x_1 = y_1 = 1$ and $x_{n+1} = \frac{x_n+2}{x_n+1}$ and $y_{n+1} = \frac{y_n^2+2}{2y_n}$ for $n = 1, 2, ...$ Prove that $y_{n+1} = x_{2^n}$ holds for $n = 0, 1, 2, ...$

3 The triangle ABC ($AB > BC$) is inscribed in the circle Ω. On the sides AB and BC, the points M and N are chosen, respectively, so that $AM = CN$, The lines MN and AC intersect at point K. Let P be the center of the inscribed circle of triangle AMK, and Q the center of the excircle of the triangle CNK tangent to side CN. Prove that the midpoint of the arc ABC of the circle Ω is equidistant from the P and Q.

– Day III

1 Let 19 integer numbers are given. Let Hamza writes on the paper the greatest common divisor for each pair of numbers. It occurs that the difference between the biggest and smallest numbers written on the paper is less than 180. Prove that not all numbers on the paper are different.

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2 Let $ABCD$ is a trapezoid with $\angle A = \angle B = 90^o$ and let E is a point lying on side CD. Let the circle ω is inscribed to triangle ABE and tangents sides AB, AE and BE at points P, F and K respectively. Let KF intersects segments BC and AD at points M and N respectively, as well as PM and PN intersect ω at points H and T respectively. Prove that $PH = PT$.

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3 Let 300 students participate to the Olympiad. Between each 3 participants there is a pair that are not friends. Hamza enumerates participants in some order and denotes by x_i the number of friends of i-th participant. It occurs that $\{x_1, x_2, ..., x_{299}, x_{300}\} = \{1, 2, ..., N - 1, N\}$ Find the biggest possible value for N.
