Saudi Arabia IMO Team Selection Test 2017

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– Day I

1 Let ABC be a triangle inscribed in circle (O), with its altitudes BE, CF intersect at orthocenter H ($E \in AC, F \in AB$). Let M be the midpoint of BC, K be the orthogonal projection of H on AM. EF intersects BC at P. Let Q be the intersection of tangent of (O) which passes through A with BC, T be the reflection of Q through P. Prove that $\angle OKT = 90^o$.

2 Find all $f : \mathbb{R} \to \mathbb{R}$ satisfying:

$$f(xf(y) - y) + f(xy - x) + f(x + y) = 2xy, \quad \forall x, y \in \mathbb{R}.$$

3 Prove that there are infinitely many positive integers n such that n divides $2017^{2017^n-1} - 1$ but n does not divide $2017^n - 1$.

– Day II

1 In the garden of Wonderland, there are 2016 apples, 2017 bananas and 2018 oranges.Two monkeys Adu and Bakar play the following game: alternatively each of them takes and eats one fruit of any kind except for the one that he took in previous turn (in the first turn, each of them can take a fruit of any kind). Who can not take a fruit is the loser. Which monkey has the winning strategy if Adu plays first?

2 Let $ABCD$ be the circumscribed quadrilateral with the incircle (I). The circle (I) touches AB, BC, CD, DA at M, N, P, Q respectively. Let K and L be the circumcenters of the triangles AMN and APQ respectively. The line KL cuts the line BD at R. The line AI cuts the line MQ at J. Prove that $RA = RJ$.

3 Find the greatest positive real number M such that for all positive real sequence (a_n) and for all real number $m < M$, it is possible to find some index $n \ge 1$ that satisfies the inequality $a_1 + a_2 + a_3 + ... + a_n + a_{n+1} > m a_n$.

– Day III

1 For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \ge 2017$, the integer $P(n)$ is positive and $S(P(n)) = P(S(n))$.

2 Let $ABCD$ be a quadrilateral inscribed a circle (O). Assume that AB and CD intersect at E, AC and BD intersect at K, and O does not belong to the line KE. Let G and H be the midpoints of AB and CD respectively. Let (I) be the circumcircle of the triangle GKH. Let (I) and (O) intersect at M, N such that $MGHN$ is convex quadrilateral. Let P be the intersection of MG and HN, Q be the intersection of MN and GH.
a) Prove that IK and OE are parallel.
b) Prove that PK is perpendicular to IQ.

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3 The 64 cells of an 8×8 chessboard have 64 different colours. A Knight stays in one cell. In each move, the Knight jumps from one cell to another cell (the 2 cells on the diagonal of an 2×3 board) also the colours of the 2 cells interchange. In the end, the Knight goes to a cell having common side with the cell it stays at first. Can it happen that: there are exactly 3 cells having the colours different from the original colours?

– Day IV

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1 Let a, b and c be positive real numbers such that min $\{ab, bc, ca\} \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq (\frac{a+b+c}{3})^2 + 1$$

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2 Denote by $\{x\}$ the fractional part of a real number x, that is $\{x\} = x - \rfloor x \lfloor$ where $\rfloor x \lfloor$ is the maximum integer not greater than$x$. Prove that
a) For every integer n, we have $\{n\sqrt{17}\} > \frac{1}{2\sqrt{17}n}$
b) The value $\frac{1}{2\sqrt{17}}$ is the largest constant c such that the inequality $\{n\sqrt{17}\} > cn$ holds for all positive integers n

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3 For integer $n > 1$, consider n cube polynomials $P_1(x), ..., P_n(x)$ such that each polynomial has 3 distinct real roots. Denote S as the set of roots of following equation $P_1(x)P_2(x)P_3(x)...P_n(x) = 0$.
It is also known that for each $1 \leq i < j \leq n, P_i(x)P_j(x) = 0$ has 5 distinct real roots.
1. Prove that if for each $a, b \in S$, there is exactly one $i \in \{1, 2, 3, ..., n\}$ such that $P_i(a) = P_i(b) = 0$ then $n = 7$.
2. Prove that if $n > 7$ then $|S| = 2n + 1$.
