Saudi Arabia IMO Team Selection Test 2019

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by parmenides51

– Day I

1 Find all functions $f : Z^+ \to Z^+$ such that $n^3 - n^2 \le f(n) \cdot (f(f(n)))^2 \le n^3 + n^2$ for every n in positive integers

2 Find all pair of integers (m, n) and $m \ge n$ such that there exist a positive integer s and
a) Product of all divisor of sm, sn are equal.
b) Number of divisors of sm, sn are equal.

3 Let regular hexagon is divided into $6n^2$ regular triangles. Let $2n$ coins are put in different triangles such, that no any two coins lie on the same layer (layer is area between two consecutive parallel lines). Let also triangles are painted like on the chess board. Prove that exactly n coins lie on black triangles.
https://cdn.artofproblemsolving.com/attachments/0/4/96503a10351b0dc38b611c6ee6eb945b5ed1c png

– Day II

1 Let a_0 be an arbitrary positive integer. Let (a_n) be infinite sequence of positive integers such that for every positive integer n, the term a_n is the smallest positive integer such that $a_0 + a_1 + ... + a_n$ is divisible by n. Prove that there exist N such that $a_{n+1} = a_n$ for all $n \ge N$

2 Let non-constant polynomial $f(x)$ with real coefficients is given with the following property: for any positive integer n and k, the value of expression

$$\frac{f(n+1)f(n+2)...f(n+k)}{f(1)f(2)...f(k)} \in Z$$

Prove that $f(x)$ is divisible by x

3 Let ABC be an acute nonisosceles triangle with incenter I and (d) is an arbitrary line tangent to (I) at K. The lines passes through I, perpendicular to IA, IB, IC cut (d) at A_1, B_1, C_1 respectively. Suppose that (d) cuts BC, CA, AB at M, N, P respectively. The lines through M, N, P and respectively parallel to the internal bisectors of A, B, C in triangle ABC meet each other to define a triange XYZ. Prove that three lines AA_1, BB_1, CC_1 are concurrent and IK is tangent to the circle (XYZ)