239 Open Mathematical Olympiad 2008

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– Grade 10-11

1 Composite numbers a and b have equal number of divisors. All proper divisors of a were written in ascending order and all proper divisors of b were written under them in ascending order, then the numbers that are below each other were added together. It turned out that the resulting numbers formed a set of all proper divisors of a certain number. What are the smallest values that a and b take?

2 A circumscribed quadrilateral $ABCD$ is given. E and F are the intersection points of opposite sides of the $ABCD$. It turned out that the radii of the inscribed circles of the triangles AEF and CEF are equal. Prove that $AC\perp BD$.

3 Prove that you can arrange arrows on the edges of a convex polyhedron such that each vertex contains at most three arrows.

4 For what natural number $n>100$ can n pairwise distinct numbers be arranged on a circle such that each number is either greater than 100 numbers following it clockwise or less than all of them? and would any property be violated when deleting any of those numbers?

5 In the triangle ABC, $\angle B=120^\circ$, point M is the midpoint of side AC. On the sides AB and BC, the points K and L are chosen such that $KL \parallel AC$. Prove that $MK+ML\geq MA$.

6 Given a polynomial $P(x,y)$ with real coefficients, suppose that some real function $f:\mathbb{R}\to\mathbb{R}$ satisfies

$$P(x,y)=f(x+y)-f(x)-f(y)$$

for all $x,y\in\mathbb{R}$. Show that some polynomial q satisfies

$$P(x,y)=q(x+y)-q(x)-q(y)$$

7 Find all natural numbers n,k such that

$$2^n5^k=7.$$

8 The natural numbers $x_1, x_2, \dots, x_n$ are such that all their 2^n partial sums are distinct. Prove that:

$$x_1{}^2 + x_2{}^2 + \dots + x_n{}^2 \geq \frac{4^n 1}{3}.$$

– Grade 8-9

1 An odd natural number k is given. Consider a composite number n. We define $d(n)$ the set of proper divisors of number n. If for some number m, $d(m)$ is equal to $d(n) \cup \{k\}$, we call n a good number. prove that there exist only finitely many good numbers.
(A proper divisor of a number is any divisor other than one and the number itself.)

2 For all positive numbers a, b, c satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, prove that:

$$\frac{a}{a+bc} + \frac{b}{b+ca} + \frac{c}{c+ab} \geq \frac{3}{4}.$$

3 A connected graph has 100 vertices, the degrees of all the vertices do not exceed 4 and no two vertices of degree 4 are adjacent. Prove that it is possible to remove several edges that have no common vertices from this graph such that there would be no triangles in the resulting graph.

4 Point P is located inside an acute-angled triangle ABC. A_1, B_1, C_1 are points symmetric to P with respect to the sides of triangle ABC. It turned out that the hexagon $AB_1CA_1BC_1$ is inscribed. Prove that P is the Torricelli point of triangle ABC.

5 You are given a checkered square, the side of which is $n1$ long and contains $n \geq 10$ nodes. A non-return path is a path along edges, the intersection of which with any horizontal or vertical line is a segment, point or empty set, and which does not pass along any edge more than once. What is the smallest number of non-return paths that can cover all the edges?
(An edge is a unit segment between adjacent nodes.)

6 AB is the chord of the circle S. Circles S_1 and S_2 touch the circle S at points P and Q, respectively, and the segment AB at point K. It turned out that $\angle PBA = \angle QBA$. Prove that AB is the diameter of the circle S.

7 Same as grade 10-11, 4

8 Same as grade 10-11, 7

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