ELMO Problems 2020

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Day	1
P1	Let $\mathbb{N}$ be the set of all positive integers. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that $$f^{f^{f(x)}(y)}(z)=x+y+z+1$$ for all $x,y,z\in\mathbb{N}$. *Proposed by William Wang.*
P2	Define the Fibonacci numbers by $F_1=F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for $n\geq 3$. Let k be a positive integer. Suppose that for every positive integer m there exists a positive integer n such that $m\mid F_n-k$. Must k be a Fibonacci number? *Proposed by Fedir Yudin.*
P3	Janabel has a device that, when given two distinct points U and V in the plane, draws the perpendicular bisector of UV. Show that if three lines forming a triangle are drawn, Janabel can mark the orthocenter of the triangle using this device, a pencil, and no other tools. *Proposed by Fedir Yudin.*
Day	2
P4	Let acute scalene triangle ABC have orthocenter H and altitude AD with D on side BC. Let M be the midpoint of side BC, and let D' be the reflection of D over M. Let P be a point on line $D'H$ such that lines AP and BC are parallel, and let the circumcircles of $\triangle AHP$ and $\triangle BHC$ meet again at $G\neq H$. Prove that $\angle MHG=90^\circ$. *Proposed by Daniel Hu.*
P5	Let m and n be positive integers. Find the smallest positive integer s for which there exists an $m\times n$ rectangular array of positive integers such that -each row contains n distinct consecutive integers in some order, -each column contains m distinct consecutive integers in some order, and -each entry is less than or equal to s. *Proposed by Ankan Bhattacharya.*

P6 For any positive integer n, let

-$\tau(n)$ denote the number of positive integer divisors of n,
-$\sigma(n)$ denote the sum of the positive integer divisors of n, and
-$\varphi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n.

Let $a, b > 1$ be integers. Brandon has a calculator with three buttons that replace the integer n currently displayed with $\tau(n)$, $\sigma(n)$, or $\varphi(n)$, respectively. Prove that if the calculator currently displays a, then Brandon can make the calculator display b after a finite (possibly empty) sequence of button presses.

Proposed by Jaedon Whyte.