IMC 2020

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–	Day 1 (July 26)
1	Let n be a positive integer. Compute the number of words w that satisfy the following three properties. 1. w consists of n letters from the alphabet $\{a, b, c, d\}$. 2. w contains an even number of a's 3. w contains an even number of b's. For example, for $n = 2$ there are 6 such words: aa, bb, cc, dd, cd, dc.
2	A, B are $n \times n$ matrices such that rank$(AB - BA + I) = 1$. Prove that tr$(ABAB) -$ tr$(A^2B^2) = \frac{1}{2}n(n-1)$.
3	Let $d \geq 2$ be an integer. Prove that there exists a constant $C(d)$ such that the following holds: For any convex polytope $K \subset \mathbb{R}^d$, which is symmetric about the origin, and any $\varepsilon \in (0, 1)$, there exists a convex polytope $L \subset \mathbb{R}^d$ with at most $C(d)\varepsilon^{1-d}$ vertices such that $$(1-\varepsilon)K \subseteq L \subseteq K.$$ Official definitions: For a real α, a set $T \in \mathbb{R}^d$ is a [i]convex polytope with at most α vertices[/i], if T is a convex hull of a set $X \in \mathbb{R}^d$ of at most α points, i.e. $T = \{\sum\limits_{x \in X} t_x x \mid t_x \geq 0, \sum\limits_{x \in X} t_x = 1\}$. Define $\alpha K = \{\alpha x \mid x \in K\}$. A set $T \in \mathbb{R}^d$ is *symmetric about the origin* if $(-1)T = T$.
4	A polynomial p with real coefficients satisfies $p(x+1) - p(x) = x^{100}$ for all $x \in \mathbb{R}$. Prove that $p(1-t) \geq p(t)$ for $0 \leq t \leq 1/2$.
–	Day 2 (July 27)
5	Find all twice continuously differentiable functions $f : \mathbb{R} \to (0, \infty)$ satisfying $f''(x)f(x) \geq 2f'(x)^2$.
6	Find all prime numbers p such that there exists a unique $a \in \mathbb{Z}_p$ for which $a^3 - 3a + 1 = 0$.

7 Let G be a group and $n \geq 2$ be an integer. Let H_1, H_2 be 2 subgroups of G that satisfy

$$[G : H_1] = [G : H_2] = n \text{ and } [G : (H_1 \cap H_2)] = n(n-1).$$

Prove that H_1, H_2 are conjugate in G.

Official definitions: $[G : H]$ denotes the index of the subgroup of H, i.e. the number of distinct left cosets xH of H in G. The subgroups H_1, H_2 are conjugate if there exists $g \in G$ such that $g^{-1}H_1g = H_2$.

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8 Compute $\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k.$
