239 Open Mathematical Olympiad 2012

www.artofproblemsolving.com/community/c1243930

by matinyousefi, SinaQane

–	Grade 10-11
1	On a 10×10 chessboard, several knights are placed, and in any 2×2 square there is at least one knight. What is the smallest number of cells these knights can threat? (The knight does not threat the square on which it stands, but it does threat the squares on which other knights are standing.)
2	Natural numbers a, b, c, d are given such that $c > b$. Prove that if $a+b+c+d = ab-cd$, then $a+c$ is a composite number.
3	There are n points in the space such that none 4 of them lie on a plane. You can select two points A and B and move point A to the midpoint of line segment AB. It turned out that, after several moves, the points took the same places (possibly in a different order). What is the smallest value of n that this could happen for some n points?
4	For some positive numbers a, b, c and d, we know that $$\frac{1}{a^3+1}+\frac{1}{b^3+1}+\frac{1}{c^3+1}+\frac{1}{d^3+1}=2.$$ Prove that $$\frac{1-a}{a^2-a+1}+\frac{1-b}{b^2-b+1}+\frac{1-c}{c^2-c+1}+\frac{1-d}{d^2-d+1} \geq 0.$$
5	Point M is the midpoint of the base AD of trapezoid $ABCD$ inscribed in circle S. Rays AB and DC intersect at point P, and ray BM intersects S at point K. The circumscribed circle of triangle PBK intersects line BC at point L. Prove that $\angle LDP = 90^\circ$.
6	In an n-element set S, several subsets $A_1, A_2, \ldots, A_k$ are distinguished, each consists of at least two, but not all elements of S. What is the largest k that its possible to write down the elements of S in a row in the order such that we dont find all of the element of an A_i set in the consecutive elements of the row?
7	Vasya conceived a two-digit number a, and Petya is trying to guess it. To do this, he tells Vasya a natural number k, and Vasya tells Petya the sum of the digits of the number ka. What is the smallest number of questions that Petya has to ask so that he can certainly be able to determine Vasyas number?

8 We call a tetrahedron divisor of a parallelepiped if the parallelepiped can be divided into 6 copies of that tetrahedron. Does there exist a parallelepiped that it has at least two different divisor tetrahedrons?

– Grade 8-9

4 For positive real numbers a, b, and c with $a+b+c=1$, prove that:

$$(a-b)^2+(b-c)^2+(c-a)^2 \geq \frac{1-27abc}{2}.$$

5 On the hypotenuse AB of the right-angled triangle ABC, a point K is chosen such that $BK = BC$. Let P be a point on the perpendicular line from point K to the line CK, equidistant from the points K and B. Also let L denote the midpoint of the segment CK. Prove that line AP is tangent to the circumcircle of the triangle BLP.

6 Let G be a planar graph all of whose vertices are of degree 4. Vasya and Petya walk along its edges. The first time each of them goes as he pleases, and then each of them goes straight (from the three roads they have to choose the middle one). As the result, each vertex was visited by exactly one of them and exactly once. Prove that this graph has an even number of vertices.

7 A circumscribed quadrilateral $ABCD$ is given. It is known that $\angle ACB = \angle ACD$. On the angle bisector of $\angle C$, a point E is marked such that $AE \perp BD$. Point F is the foot of the perpendicular line from point E to the side BC. Prove that $AB = BF$.

– Thanks should go to Alireza Danaie for translation.