239 Open Mathematical Olympiad 2010

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– Grade 10-11

1 Each square of the chessboard was painted in one of eight colors in such a way that the number of squares colored by all the colors are equal. Is it always possible to put 8 rooks not threatening each other on multi-colored cells?

2 The incircle of the triangle ABC touches the sides AC and BC at points K and L, respectively. the B-excircle touches the side AC of this triangle at point P. The segment AL intersects the inscribed circle for the second time at point S. Line KL intersects the circumscribed circle of triangle ASK for the second at point M. Prove that $PL = PM$.

3 Grisha wrote n different natural numbers, the sum of which does not exceed S. The saboteur added to each of them a number from the half-interval $[0, 1)$. The sabotage is successful if there exists two subsets, the sums of the numbers in which differ by no more than 1. At what minimum S can Grisha ensure that the sabotage will definitely not be succeeded?

4 Consider three pairwise intersecting circles ω_1, ω_2 and ω_3. Let their three common chords intersect at point R. We denote by O_1 the center of the circumcircle of a triangle formed by some triple common points of ω_1 & ω_2, ω_2 & ω_3 and ω_3 & ω_1. and we denote by O_2 the center of the circumcircle of the triangle formed by the second intersection points of the same pairs of circles. Prove that points R, O_1 and O_2 are collinear.

5 Given three natural numbers greater than 100, that are pairwise coprime and such that the square of the difference of any two of them is divisible by the third and any of them is less than the product of the other two. Prove that these numbers are squares of some natural numbers.

6 We have six positive numbers $a_1, a_2, \dots, a_6$ such that $a_1a_2\dots a_6 = 1$. Prove that:

$$\frac{1}{a_1(a_2+1)} + \frac{1}{a_2(a_3+1)} + \dots + \frac{1}{a_6(a_1+1)} \geq 3.$$

7 You are given a convex polygon with perimeter $24\sqrt{3}+4\pi$. If there exists a pair of points dividing the perimeter in half such that the distance between them is equal to 24, Prove that there exists a pair of points dividing the perimeter in half such that the distance between them does not exceed 12.

8 Consider the graph G with 100 vertices, and the minimum odd cycle goes through 13 vertices. Prove that the vertices of the graph can be colored in 6 colors in a way that no two adjacent vertices have the same color.

– Grade 8-9

2 The incircle of the triangle ABC touches the sides AC and BC at points K and L, respectively. the B-excircle touches the side AC of this triangle at point P. Line KL intersects with the line passing through A and parallel to BC at point M. Prove that $PL = PM$.

3 Same as grade 10-11, 1

4 Same as grade 10-11, 3

5 Among 33 balls, there are 2 radioactive ones. You can put several balls in the detector and it will show if the both radioactive balls are among the balls. What is the smallest number that we have to use the detector so that one can certainly find at least one of the radioactive balls?

6 We call natural numbers n and k are similar if they are multiples of square of a number greater than 1. Let $f(n)$ denote the number of numbers from 1 to n similar to n (for example, $f(16) = 4$, since the number 16 is similar to 4, 8, 12 and 16). What integer values can the quotient $\frac{n}{f(n)}$ take?

7 In a convex quadrilateral $ABCD$, We have $\angle B = \angle D = 120^\circ$. Points A', B' and C' are symmetric to D relative to BC, CA and AB, respectively. Prove that lines AA', BB' and CC' are concurrent.

8 For positive numbers x, y, and z, we know that $x + y^2 + z^3 = 1$. Prove that

$$\frac{x}{2+xy} + \frac{y}{2+yz} + \frac{z}{2+zx} > \frac{1}{2}.$$

– Thanks should go to Alireza Danaie for translation.