239 Open Mathematical Olympiad 2009

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–	Grade 10-11
1	In a sequence of natural numbers, the first number is a, and each subsequent number is the smallest number coprime to all the previous ones and greater than all of them. Prove that in this sequence from some place all numbers will be primes.
2	On the sides AB, BC and CA of triangle ABC, points K, L and M are selected, respectively, such that $AK = AM$ and $BK = BL$. If $\angle MLB = \angle CAB$, Prove that $ML = KI$, where I is the incenter of triangle CML.
3	200 sticks are given whose lengths are $1, 2, 4, \ldots, 2^{199}$. What is the smallest number of sticks needed to be broken so that out of all the resulting sticks, several triangles could be created, if each stick could be broken only once, and each triangle can be created out of only three sticks?
6	Non-negative integers are placed on the vertices of a 100-gon, the sum of the numbers is 99. Every minute at one of the vertices that is equal to 0 will be replaced by 2 and both its neighboring numbers are subtracted by 1. Prove that after a while a negative number will appear on the board.
4	Natural numbers a and b are given such that the number $$P = \frac{[a,b]}{a+1} + \frac{[a,b]}{b+1}$$ Is a prime. Prove that $4P+5$ is the square of a natural number.
7	The Feuerbach point (the tangent point of the inscribed circle and the nine-point circle of triangle ABC) F is marked in triangle ABC. A_1 is on the side BC such that AA_1 is the altitude of triangle ABC. Prove that the line symmetric to FA_1 with respect to BC is perpendicular to IO, where O is the center of the circumcircle of the triangle ABC and I is the center of its incircle.
8	Alireza multiplied a billion consecutive natural numbers, and Matin multiplied two million consecutive natural numbers. Prove that these two got different results or one of them has made a mistake.
–	Grade 8-9

1 Kostya drives a car from a village to a city, driving along three roads. Moreover, on each of these roads, he drives at a constant speed. Is it possible that a third of the distance traveled was completed earlier than a third of the time, half of the distance traveled later than half of the time, and two-thirds of the distance was earlier than two-thirds of the time?

2 Same as grade 10-11, 1

3 The company has 100 people. For any k, we can find a group of k people such that there are two (different from them) strangers, each of them knows all of these k people. At what maximum k is this possible?

4 The natural numbers $x, y > 1$, are such that $x^2 + xy - y$ is the square of a natural number. Prove that $x + y + 1$ is a composite number.

5 Same as grade 10-11, 2

6 Same as grade 10-11, 6

7 In the triangle ABC, the cevians AA_1, BB_1 and CC_1 intersect at the point O. It turned out that AA_1 is the bisector, and the point O is closer to the straight line AB than to the straight lines A_1C_1 and B_1A_1. Prove that $\angle BAC > 120^\circ$.

8 Each of the 11 girls wants to mail each of the other a gift for Christmas. The packages contain no more than two gifts. If they have enough time, what is the smallest possible number of packages that they have to send?

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