239 Open Mathematical Olympiad 2019

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–   Grade 10-11

1   The following fractions are written on the board $\frac{1}{n}, \frac{2}{n-1}, \frac{3}{n-2}, \dots, \frac{n}{1}$ where n is a natural number. Vasya calculated the differences of the neighboring fractions in this row and found among them 10000 fractions of type $\frac{1}{k}$ (with natural k). Prove that he can find even 5000 more of such these differences.

2   Several cells are marked in a 100×100 table. Vasya wants to split the square into several rectangles such that each rectangle does not contain more than two marked cells and there are at most k rectangles containing less than two cells. What is the smallest k such that Vasya will certainly be able to do this?

3   The radius of the circumscribed circle of an acute-angled triangle is 23 and the radius of its Inscribed circle is 9. Common external tangents to its ex-circles, other than straight lines containing the sides of the original triangle, form a triangle. Find the radius of its inscribed circle.

4   There are $n > 1000$ people at a round table. Some of them are knights who always tell the truth, and the rest are liars who always tell lies. Each of those sitting said the phrase: among the 20 people sitting clockwise from where I sit there are as many knights as among the 20 people seated counterclockwise from where I sit. For what n could this happen?

5   Circle Γ touches the circumcircle of triangle ABC at point R, and it touches the sides AB and AC at points P and Q, respectively. Rays PQ and BC intersect at point X. The tangent line at point R to the circle Γ meets the segment QX at point Y. The line segment AX intersects the circumcircle of triangle APQ at point Z. Prove that the circumscribed circles of triangles ABC and XYZ are tangent.

6   Find all functions $f : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions: (i) $f(x) + f(\frac{1}{x}) = 1$ for all $x > 0$; (ii) $f(xy + x + y) = f(x)f(y)$ for all $x, y > 0$.

7   Given positive numbers $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n$. Let m_k be the maximum of the products $a_i b_j c_l$ over the sets (i, j, l) for which $max(i, j, l) = k$. Prove that

$$(a_1 + \dots + a_n)(b_1 + \dots + b_n)(c_1 + \dots + c_n) \le n^2(m_1 + \dots + m_n).$$

8 Given a natural number $k > 1$. Prove that if through any edge of the graph G passes less than $[e(k-1)! - 1]$ simple cycles, then the vertices of this graph can be colored with k colors in the correct way.

– Grade 8-9

1 On the island of knights and liars, a tennis tournament was held, in which 100 people participated in. Each two of them played exactly 1 time with the other one. After the tournament, each of the participants declared: I have beaten as many knights as liars, while all the knights told the truth, and all the liars lied. What is the largest number of knights that could participate in the tournament?

2 Is it true that there are 130 consecutive natural numbers, such that each of them has exactly 900 natural divisors?

3 Circle ω touches the side AC of the equilateral triangle ABC at point D, and its circumcircle at the point E lying on the arc $\overset{\frown}{BC}$. Prove that with segments AD, BE and CD, you can form a triangle, in which the difference of two of its angles is $60°$.

4 A 20×20 treasure map is glued to a torus. A treasure is hidden in a cell of this map. We can ask questions about 1×4 or 4×1 rectangles so that we find out if there is a treasure in this rectangle or not. The answers to all questions are absolutely true, but they are given only after all rectangles we want to ask are set. What is the least amount of questions needed to be asked so that we can be sure to find the treasure? (If you describe the position of the cells in a torus with numbers (i,j) of row and column, $1 \le i, j \le 20$, then two cells are neighbors, if and only if two of the coordinates they have are the same, and the other two differ by 1 mod 20.)

5 We call an ordered set of distinct natural numbers good if for any two numbers in it, the larger one is divided by the smaller one. Prove that the number $(n+1)!1$ can be represented as $x_1 + 2x_2 + \ldots + nx_n$, where $\{x_1, x_2, \ldots, x_n\}$ is a good set, by at least $n!$ ways.

6 Same as grade 10-11, 3

7 Same as grade 10-11, 7

8 There are n instruments in the laboratory, each two of them can be connected with a wire. Moreover, if four devices A, B, C, D, are such that wires of AB, BC and CD are connected but there is no connected pair between CA, AD and DB, a collapse occurs. A professor invented a wiring diagram that does not collapse. Coming to the laboratory, he found that the collapse has not yet occurred, but the devices are connected not according to his scheme. Prove that he can implement his scheme, each time connecting or disconnecting a pair of devices, so that the collapse wont happen anytime.

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